J. Fluid Mech. (1975), *vol.* 69, *part* 1, *pp.* 145–177
Printed in Great Britain

Random waves and dynamo action

By A. M. SOWARD
School of Mathematics, University of Newcastle upon Tyne, England

(Received 26 November 1973 and in revised form 22 June 1974)

The propagation of waves in an inviscid, electrically conducting fluid is considered. The fluid rotates with angular velocity Ω^* and is permeated by a magnetic field $\mathbf{b}^*$ which varies on the length scale $L = Ql$, where $Q = \Omega^* l^2/\lambda$ (l is the length scale of the waves, λ is the magnetic diffusivity) is assumed large ($Q \gg 1$). A linearized theory is readily justified in the limit of zero Rossby number R_0 ($= U_0/\Omega^* l$, where U_0 is a typical fluid velocity) and for this case it is shown that the total wave energy of a wave train is conserved and transported at the group velocity except for that which is lost by ohmic dissipation. The analysis is extended to encompass the propagation of a sea of random waves.

A hydromagnetic dynamo model is considered in which the fluid is confined between two horizontal planes perpendicular to the rotation axis a distance L_0 ($= O(L)$) apart. Waves of given low frequency ω_0^* ($= O(R_0 Q^{\frac{1}{2}} \Omega^*)$) and horizontal wavenumber l^{-1} but random orientation are excited at the lower boundary, where the kinetic energy density is $2\pi\rho U_0^2$. The waves are absorbed perfectly at the upper boundary, so that there is no reflexion. The linear wave energy equation remains valid in the double limit $1 \gg R_0 Q^{\frac{1}{2}} \gg Q^{-\frac{1}{2}}$, for which it is shown that dynamo action is possible provided $\Delta = L_0 U_0^2/l^3 \omega_0^{*2} > 1$. When dynamo action maintains a weak magnetic field ($\Delta - 1 \ll 1$) which only slightly modifies the inertial waves analytic solutions are obtained. In the case of a strong magnetic field ($\Delta \gg 1$) for which Coriolis and Lorentz forces are comparable solutions are obtained numerically. The latter class includes the more realistic case ($\Delta \to \infty$) in which the upper boundary is absent.

1. Introduction

An interesting approach to the theory of dynamo action in an inviscid incompressible fluid rotating with angular velocity Ω^* has been presented by Moffatt (1970, 1972; henceforth referred to as I, II). In I a sea of random inertial waves is considered. The waves can be divided into two classes: the waves with a positive (negative) component of group velocity in the direction of rotation are referred to as upward (downward) propagating waves. It is supposed that only upward-propagating waves are present and that these are distributed both isotropically and homogeneously. A small seed magnetic field is introduced having a length scale L which is large compared with the length scale l of the inertial waves. While the magnetic field is sufficiently weak to have no influence on the development of the inertial waves, the energy spectrum tensor retains its initial value. The α-effect can therefore be computed and the resulting kinematic dynamo problem is readily solved. It is found that provided the length scale of the magnetic field is sufficiently large magnetic-field regeneration is always possible. There is, however, a fastest-growing mode identical in structure to the magnetic field defined by equation (6.6 a) below which dominates after a long time. For obvious reasons the dominant mode of linear theory is selected in the subsequent nonlinear stage in the development of the system, i.e. when the finite Lorentz force is taken into account. Because of the simple form of the magnetic field, this latter stage is also readily treated.

In II a random force field which excites waves at their natural frequency is assumed to prevail. With the introduction of viscous diffusion in addition to magnetic diffusion the waves have finite amplitude and do not grow indefinitely. The assumed properties of the random body force have similar characteristics to the sea of inertial waves postulated in I. Consequently only upward-propagating waves are investigated. As in I the α-effect is calculated and the ultimate steady state of the hydromagnetic dynamo is determined.

The analysis of waves in a uniform medium is usually initiated by seeking wavelike solutions of the form $\mathbf{u}^* \exp i(\mathbf{k}^*.\mathbf{x}^* - \omega^* t^*)$, where $\mathbf{u}^*$, $\mathbf{k}^*$ and ω^* are constants, $\mathbf{x}^*$ is the position and t^* is the time. Then, for given wave vector $\mathbf{k}^*$, the frequency ω^* is determined by a dispersion relation. The waves in I are treated on this basis with the assumption that the magnetic field is constant. Since there is ohmic dissipation the waves decay at a slow rate determined by the negative imaginary part of ω^* (say $-i\omega_i^*$). Consequently the energy spectrum becomes $e^*(\mathbf{k}^*) \exp(-2\omega_i^* t^*)$, where $e^*(\mathbf{k}^*)$ is its initial value. Though this treatment of the waves is satisfactory in a first approximation this is not necessarily the case at higher orders when the slow evolution of the energy spectrum is investigated. For then it must be anticipated that the amplitude of the waves will vary on the same length scale as the magnetic field. Indeed it may be shown that the wave energy spectrum $e^*(\mathbf{k}^*; \mathbf{x}^*, t^*)$ is constant at points moving with the group velocity except for that lost by ohmic dissipation [see (2.32) and (C 3) below]. Here $\mathbf{k}^*$ is the local wave vector, which evolves according to differential equations similar to (2.10) and (2.20) below. To solve the hydromagnetic dynamo problem the equation for the energy spectrum and the induction equation governing the mean magnetic field must be solved simultaneously. In the parameter range for which magnetic-field regeneration is considered in I it is shown (see appendix C) that advection at the group velocity dominates any local rate of change, so that the form assumed for the energy spectrum is too simplistic. In the same range of parameters a further complication results from nonlinear wave interactions. This effect introduces further nonlinear terms in the energy equation (2.32) below which may have a significant effect over the dynamo time scale (see appendix C).

The purpose of the present paper is twofold. First the analysis is developed to obtain new equations for the hydromagnetic dynamo. To this end a single wave train is considered and the equation (2.32) which governs the evolution of the total wave energy $E^*(\mathbf{k}^*)$ is derived. Similar equations have been postulated previously in the linearized theory of wave propagation for non-conservative systems in which the background medium varies slowly (e.g. see Bretherton

1970; Landahl 1972). The author is, however, unaware of any formal derivation of an equation similar to (2.32). Consequently, even though (2.32) is obtained in the restricted context of large-scale magnetic-field variations and ohmic decay, the result is perhaps of more general interest.

Since the treatment of a single wave train involves only one wave vector $\mathbf{k}^*$ at any instant in space and time, the analysis is extended in §3 to encompass a continuum of wave vectors and corresponding wave trains. The concept appears novel but only by this device is it possible to discuss the propagation of truly random waves in a medium with slowly varying properties. The extension is rendered non-trivial solely by the property† that, for a given wave train, $\mathbf{k}^*$ does not necessarily remain fixed. The analysis reveals, however, that the total wave energy spectrum e^* in $\mathbf{k}^*$ space is related simply to E^* by a weighting factor (or density) σ^*. Here σ^* accounts for variations in the volume d^3k^* occupied by a wave train whose total energy is $E^*\sigma^*d^3k^*$, where the factor $\sigma^*d^3k^*$ remains fixed as $\mathbf{k}^*$ varies. Though the full development of §3 is not required in the later sections it is appropriate to formulate the problem in its entirety as it provides the key step in the extension of mean-field electrodynamics to the dynamics of the medium.

To place hydromagnetic dynamos based on wave motions in perspective, it must be appreciated that only two hydromagnetic dynamos investigated in detail take account of the nature of the energy source. Busse (1973) has proposed a model based on Bénard convection combined with a shear flow, while Childress & Soward (1972) have developed a model in which rotation has a controlling influence. In both convection-driven models quasi-steady flows are envisaged, though wave motions are possible in the latter model if the Prandtl number is sufficiently small. The forced excitation of waves could then ensue in the form of overstability (e.g. see Chandrasekhar 1961, p. 118). The possible importance of wave motions in the dynamo process has long been appreciated. Braginskiĭ (1964*c*) has given a comprehensive survey of possible wave excitation in the earth's core by buoyancy forces and more recently (1967) has investigated non-dissipative waves in which magnetic, Archimedean (buoyancy) and Coriolis forces all play a significant role: the so-called MAC waves. The instability envisaged to drive the MAC waves is of a faster type than the slow resistive instability mentioned above, where dissipation plays a key role.‡ So far no dynamo sustained by wave motions, which incorporates the above ideas, has received detailed treatment. Indeed the pioneering work in I bypasses completely the need to maintain the waves by an external force field such as gravity. This happy state of affairs is achieved through the large store of kinetic energy in the inertial waves. Thus, while magnetic field is regenerated, kinetic energy is continually

† For fully turbulent flows various techniques have been developed to take account of large-scale spatial variations of the mean magnetic field (Gubbins 1974) and turbulent intensity (Roberts & Soward 1975). Unfortunately the procedures depend on expansions of smoothly varying functions in $\mathbf{k}^*$, ω^* space and are inappropriate for random waves since they fail to accommodate the special property that wave energy is localized close to a three-dimensional manifold lying within the four-dimensional $\mathbf{k}^*$, ω^* space.

‡ For a more detailed discussion on buoyancy-driven waves, see Soward & Roberts (1975).

converted into magnetic energy: equipartition is never reached. Again, in II a random body force is invoked so that steady dynamo action is possible. In both I and II a preferred direction of wave propagation is necessary for the α-effect to operate. It is difficult to envisage how such a preferred direction could arise in a contained non-dissipative system, since waves would be reflected at the boundaries. For this reason Gubbins (1974) has pointed out that, even if there is no preferred direction of wave propagation, additional anisotropy may result from gradients of the magnetic field. Consequently he proposes that the $\boldsymbol{\omega} \times \mathbf{j}$-effect due to Rädler (1968) may be significant.

In spirit the present paper extends the attitude adopted in I, namely the complete neglect of body forces in the fluid, and follows up the suggestion that energy for the waves may be introduced by an external source. In other words an energy flux into the system at a rigid boundary (or possibly fluid interface) is conceived which can ultimately sustain a quasi-steady dynamo. The precise nature of the energy source is outside the scope of this paper and in this respect we proceed no further than previous authors. An ideal model is proposed, however, which isolates new effects. Waves are excited randomly on a plane boundary normal to the rotation axis and propagate into the fluid, where some attenuation occurs owing to weak ohmic decay. Here the large-scale magnetic field lies in planes parallel to the boundary and is itself sustained by dynamo action. The dissipative process is crucial for, even if waves are reflected from some upper boundary, there will be a net upward flux of wave energy as envisaged in I. The propagation and attenuation of wave energy is described mathematically by either the wave energy equation (2.32) for a wave train or (3.13) for random waves; the dynamo process is described by (4.1).

Though energy transport and spatial inhomogeneity should be incorporated into the Moffatt model (see appendix C) the two notions are made more transparent in the new model, where the former is necessary for fluid motions even to occur and the latter is an obvious consequence of spatial wave attenuation. Thus it provides an interesting and novel application of the wave energy equation to the hydromagnetic dynamo problem and, moreover, has certain features which make it amenable to correct mathematical treatment. In particular it is shown in §4 that the neglect of nonlinear wave interactions is justified (a technical difficulty which appears insurmountable in the Moffatt model; see appendix C) if

$$1 \gg R_0 Q^{\frac{1}{2}} \gg Q^{-\frac{1}{2}} \tag{1.1}$$

(the Rossby number R_0 and Q, a measure of the electrical conductivity, are defined by (2.3) and (2.6) below) and if the wave energy flux into the fluid is predominantly due to low frequency modes with a time scale of order

$$(R_0 Q^{\frac{1}{2}} \Omega^*)^{-1}.$$

Since the restriction to low frequency modes is crucial to the arguments of §4 justifying the linearization, it is perhaps worth noting that Braginskiĭ (1967) has investigated similar modes in some detail by the WKB method, which itself forms the basis of the perturbation procedure adopted here in §2. More significantly, when the magnetic field is weak the restriction to low frequency modes is reasonable since it isolates the most potent waves for magnetic-field regeneration. For this reason such modes played a key role in both I and II. In the weak-field case the kinetic energy $\phi^{(0)sn}$ in (2.37*b*) below remains almost constant, being virtually uninfluenced by the magnetic field. Consequently within the framework of the approximations made the α-effect, upon which generation of mean magnetic field depends, is proportional to the inverse square of the frequency [see (4.7) below]. The argument is less forceful when Lorentz forces are significant, for then $\phi^{(0)sn}$ itself depends on the frequency. Indeed, at very low frequency, significant attenuation of $\phi^{(0)sn}$ occurs on the length scale L_M [see (5.8*b*) below] which is directly proportional to the square of the frequency. The subsequent analysis is simplified by the assumption that the component l^{-1} of the wave vector perpendicular to the rotation axis and the frequency ω_0^* are fixed. The randomness is restricted to the orientation of the wave vector. The important restriction on the frequency avoids possible difficulties resulting from non-uniformities which occur in the continuous case as the excitation frequency approaches zero.

It transpires that it is expedient to introduce a second boundary into the problem. In particular the fluid is supposed to lie between the planes $z^* = 0$ and $z^* = L_0$ $(= O(Ql))$, which are normal to the axis of rotation. Whereas the waves are emitted from $z^* = 0$ they are assumed to be absorbed perfectly at $z^* = L_0$. In other words waves are not reflected and consequently only upward-propagating waves prevail. The presence of the second boundary is important as it imposes the length scale Ql on the system together with the time scale Q/Ω^*, namely the time taken for wave energy to be transported across the gap. Though the time scale Q/Ω^* is long enough for significant attenuation of the waves to occur, it is too short for nonlinear wave interactions to transfer an appreciable amount of energy into new modes. It is perhaps in this respect that the most important mathematical simplification is achieved over the model considered in I (see appendix C). Since the mean magnetic field is maintained by dynamo action it also varies on the length scale Ql. Indeed it is shown analytically that in the weak-field limit there is a minimum value of the wave energy flux $\mathbf{F}_0$ [see (7.10) below], proportional to L_0^{-1} [$\Delta = 1$ in (5.13) below], at $z^* = 0$ for which dynamo action is possible. The fact that no wave energy flux is required when $L_0 \to \infty$ is, of course, consistent with the usual state of affairs, in which dynamo action based on a kinematic theory is nearly always possible provided that the length scale of the mean magnetic field is sufficiently large. When the magnetic field is strong ($\Delta > 1$), a class of periodic solutions is determined numerically which is the extension (consequently the most likely solutions) of the mode most readily excited in the linear theory into the nonlinear regime. It is shown that in the limit $L_0 \to \infty$ (or equivalently in the absence of the second boundary) the total magnetic energy density $\mathscr{M}$ per unit area in the horizontal plane [see (5.15) below] is finite and that the magnetic field is confined to a length scale L_M [see (5.8*b*) below] inversely proportional to the wave energy flux $\mathbf{F}_0$. The quantitative nature of the results provides the most important conclusions to be drawn from the model, not because of its physical characteristics but because the introduction of the second boundary is primarily an artificial device to facilitate the mathematics. Indeed any major conclusions that depend critically on the presence of a boundary with such special properties must be regarded with suspicion. It must be re-emphasized here that the preferred direction of energy flux, on which the α-effect depends, results from wave attenuation and not the reflexional properties of the second boundary. Thus in a spherical system our analysis may have relevance locally near the boundary but then the neighbouring boundary invoked above is inappropriate.

Finally, it should be noted that the simplified geometry adopted here does not lead to certain effects that must be expected in a contained rotating system (e.g. see Greenspan 1968). Indeed the mean state is unlikely to be static as assumed in this paper: a geostrophic flow is almost inevitable, a magnetostatic balance in which Lorentz forces balance pressure forces might not be achieved and nonlinear wave interactions may excite mean flows. The last mechanism does not operate in our model (or in II) until a high order (see footnote in §7), nor does it in the case of the confined non-magnetic system discussed by Greenspan (1969). Clearly the increased complications of the contained system are outside the scope of this paper but are obviously important considerations for future studies.

2. The energy and mean-field equations

An unbounded, electrically conducting, incompressible, inviscid fluid rotates as a solid body with angular velocity $\mathbf{\Omega}^*$ and is permeated by a large-scale magnetic field $(\rho\mu)^{\frac{1}{2}}\,\Omega^* l\mathbf{B}$, where ρ is the density, μ is the magnetic permeability and the dimensionless vector $\mathbf{B}$ varies over a long length scale L ($\gg l$). Small amplitude disturbances with length scale l are superimposed on the system and propagate as waves owing to the presence of rotation and magnetic field at frequencies order Ω^*. If the fluid velocity is written as

$$\mathbf{u}^* = U_0 \mathbf{u}(\mathbf{x}, t), \tag{2.1}$$

where

$$\mathbf{x}^* = l\mathbf{x}, \quad t^* = t/\Omega^* \tag{2.2}$$

and U_0 is a typical fluid velocity ($\mathbf{u}$, $\mathbf{x}$ and t are dimensionless), the importance of convection of momentum may be measured by the Rossby number

$$R_0 = U_0/\Omega^* l. \tag{2.3}$$

Provided that the Rossby number is small,

$$R_0 \ll 1, \tag{2.4}$$

the resulting perturbation $(\rho\mu)^{\frac{1}{2}}\,U_0\mathbf{b}$ to the magnetic field is small also and the total magnetic field becomes

$$\mathbf{b}^* = (\rho\mu)^{\frac{1}{2}}\,(\Omega^* l)\,(\mathbf{B} + R_0\mathbf{b}). \tag{2.5}$$

A convenient measure of the electrical conductivity is

$$Q = \Omega^* l^2/\lambda, \tag{2.6}$$

where λ is the magnetic diffusivity. It is supposed that ohmic dissipation only causes weak attenuation of the waves and this is guaranteed by the assumption

$$Q \gg 1. \tag{2.7}$$

Since the effects of viscosity and compressibility are neglected, R_0 and Q are the only dimensionless parameters which characterize the governing equations.†

Following the usual practice in mean-field electrodynamics, the equations governing the flow and magnetic field are separated into two parts. First, equations governing the slowly varying quantities such as the mean magnetic field $\mathbf{B}$ are extracted by averaging the full equations over the large scale L. The part of the governing equations left after the mean part, symbolized by $\langle\,\rangle$, is removed determines the evolution of the fluctuating (or perturbation) quantities $\mathbf{u}$ and $\mathbf{b}$, which vary on the small scale l. If, moreover, it is supposed that Q is large but finite and that R_0 is negligible ($R_0 \to 0$) the equations governing the perturbation quantities may be linearized to give

$$\partial \mathbf{u}/\partial t + 2\mathbf{\Omega} \times \mathbf{u} = -\nabla p + (\mathbf{B}.\nabla)\,\mathbf{b} + (\mathbf{b}.\nabla)\,\mathbf{B}, \tag{2.8a}$$

$$\partial \mathbf{b}/\partial t + (\mathbf{u}.\nabla)\,\mathbf{B} = (\mathbf{B}.\nabla)\,\mathbf{u} + Q^{-1}\nabla^2\mathbf{b}, \tag{2.8b}$$

$$\nabla.\mathbf{b} = \nabla.\mathbf{u} = 0, \tag{2.8c, d}$$

where p is the modified pressure and $\mathbf{\Omega}$ is a unit vector in the z direction (z measures distance parallel to the rotation vector). The omission here of terms like $(\mathbf{u}.\nabla)\,\mathbf{u} - \langle(\mathbf{u}.\nabla)\,\mathbf{u}\rangle$ is frequently made in dynamo theory and is sometimes referred to as first-order smoothing.

For the maintenance of a hydromagnetic dynamo the above limiting procedure is too crude since the vital inductive term $\nabla \times \langle \mathbf{u} \times \mathbf{b}\rangle$ required to regenerate mean magnetic field vanishes in the limit $R_0 \to 0$ with Q fixed! Dynamo action is possible only if the induction term remains comparable in magnitude with the diffusion term. It transpires [see (4.5) below] that the balance is achieved under the subtler limiting procedure $R_0 \to 0$ with $R_0 Q^{\frac{1}{2}}$ fixed, for which the magnetic diffusivity tends to zero together with the vigour of the disturbance. In this limit, however, the linearized equations (2.8) remain valid only on making further approximations based on the assumption that $R_0 Q^{\frac{1}{2}}$ is small.

The remainder of this section and §3 are devoted entirely to the problem of linearized wave propagation in a dispersive dissipative medium. Magnetic-field regeneration is not discussed, so that the complicated approximations described above are unnecessary. For the present it is sufficient, therefore, to consider the more transparent limiting procedure,

$$R_0 \to 0 \quad (\text{fixed } Q \gg 1), \tag{2.9}$$

which leads immediately to the equations (2.8) governing the waves. The applicability of the results to the parameter range required by dynamo theory is discussed in detail in §4.

The key ideas that lie behind the energy transport by wave trains can be illustrated by the simple case in which the magnetic field $\mathbf{B}$ is both steady and uniform. Since (2.8) are now linear equations with constant coefficients, wavelike solutions proportional to $\exp[i(\mathbf{k}.\mathbf{x} - \omega t)]$ may be sought. When $Q^{-1} = 0$, four distinct waves are determined by the dispersion relation

$$D^s(\mathbf{k}, \omega) \equiv \omega^2 - 2s(\hat{\mathbf{k}}.\mathbf{\Omega})\,\omega - (\mathbf{k}.\mathbf{B})^2 = 0 \quad (s = \pm 1) \tag{2.10}$$

† Henceforth dimensionless variables are generally used, so that the superscript * is normally reserved to specify a function's complex conjugate.

(Lehnert 1954), and are defined by the frequencies

$$\omega = s\omega_n \quad (s = \pm 1), \tag{2.11a}$$

$$\omega_n = (\hat{\mathbf{k}}.\mathbf{\Omega}) + n\{(\hat{\mathbf{k}}.\mathbf{\Omega})^2 + (\mathbf{k}.\mathbf{B})^2\}^{\frac{1}{2}} \quad (n = \pm 1), \tag{2.11b}$$

where $\hat{\mathbf{k}} = \mathbf{k}/k$. When Q^{-1} is small but non-zero, the waves have the slow decay rate $Q^{-1}\omega_i$, where $\omega = \omega_r - iQ^{-1}\omega_i$ is determined by

$$D^s(\mathbf{k}, \omega; Q^{-1}) \equiv \omega^2 - 2s(\hat{\mathbf{k}}.\mathbf{\Omega})\,\omega - (\mathbf{k}.\mathbf{B})^2(1 + iQ^{-1}k^2/\omega)^{-1} = 0 \quad (s = \pm 1). \tag{2.12}$$

Thus, provided that ω and $\mathbf{k}$ are related by the linearized dispersion relation (2.10),† ω_i is given correct to order Q^{-1} by the equation

$$-i\omega_i\,\partial D^s/\partial\omega + \partial D^s/\partial Q^{-1} = 0. \tag{2.13}$$

If, instead of a single wave, a wave train is investigated, it is necessary to consider a band of wave vectors $\mathbf{k}+\mathbf{k}'$ and frequencies $\omega+\omega'$ close to some given pair $(\mathbf{k}, \omega)$ which satisfies (2.10). Expanding (2.12) as a Taylor series and retaining only the lowest-order terms gives

$$\omega'\frac{\partial D^s}{\partial\omega} + \mathbf{k}'.\frac{\partial D^s}{\partial\mathbf{k}'} + Q^{-1}\frac{\partial D^s}{\partial Q^{-1}} = 0. \tag{2.14}$$

Hence (2.13) and (2.14) together with the definition

$$\mathbf{c}_g = \partial\omega/\partial\mathbf{k} \tag{2.15}$$

for the group velocity lead to the expression

$$-i\omega' + i\mathbf{k}'.\mathbf{c}_g = -Q^{-1}\omega_i. \tag{2.16}$$

Corresponding to the four frequencies (2.11), there are four group velocities

$$\mathbf{c}_g = s\mathbf{c}_g^n \quad (s = \pm 1), \tag{2.17a}$$

where $$\mathbf{c}_g^n = \left\{-\frac{2}{k}\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{\Omega}) + 2\frac{\mathbf{k}.\mathbf{B}}{\omega_n}\mathbf{B}\right\}\Big/\left\{1 + \left(\frac{\mathbf{k}.\mathbf{B}}{\omega_n}\right)^2\right\} \quad (n = \pm 1), \tag{2.17b}$$

but there are only two decay rates

$$Q^{-1}\omega_i = \frac{k^2}{Q}\frac{(\mathbf{k}.\mathbf{B}/\omega_n)^2}{1 + (\mathbf{k}.\mathbf{B}/\omega_n)^2} \quad (n = \pm 1). \tag{2.18}$$

Provided that the wave train is modulated over the long length scale Ql and long time scale Q/Ω^*, both ω' and $\mathbf{k}'$ are of order Q^{-1} and all terms in (2.16) are comparable in magnitude. After the change of variables

$$\mathbf{X} = Q^{-1}\mathbf{x} \quad \text{and} \quad \tau = Q^{-1}t$$

it can be argued from (2.16) (see, for example, Kawahara 1973) that the wave energy is governed by equation (2.32) below. Thus the terms on the left side of (2.16) correspond to convection of wave energy at the group velocity and the term on the right side corresponds to ohmic decay. Dispersion (or diffusive

† There should be no serious confusion between the two definitions (2.10) and (2.12) for D^s, as in all subsequent applications D^s and its partial derivatives are all evaluated at $Q^{-1} = 0$.

spread) of wave energy can be traced to the order-Q^{-2} terms in the Taylor expansion (2.14). It is, therefore, a higher-order effect which is not accounted for by (2.16). Though the effect may be important in certain circumstances it is unimportant in the dynamo model investigated in §§5–7. Consequently the effects of dispersion are pursued no further.

When the large-scale magnetic field varies on the length scale $L = Ql$ the above arguments are no longer valid and the analysis must proceed more cautiously. As an example of their inadequacy, it may be noted that it is impossible to distinguish $\nabla.(\mathbf{c}_g E)$ from $(\mathbf{c}_g.\nabla)E$ in (2.32) by means of the dispersion relation alone. Since $\mathbf{c}_g$ is now a function of position, the two expressions describe different physical mechanisms and consequently the distinction becomes important.

The analysis proceeds on the basis of the formal expansions

$$\mathbf{u} = \{\mathbf{u}^{(0)}(\mathbf{X},\tau)\, e^{i\theta(\mathbf{x},t)} + Q^{-1}\mathbf{u}^{(2)}(\mathbf{X},\tau)\, e^{i\theta(\mathbf{x},t)} + \ldots\} + \text{c.c.}, \tag{2.19a}$$

$$\mathbf{b} = \{\mathbf{b}^{(0)}(\mathbf{X},\tau)\, e^{i\theta(\mathbf{x},t)} + Q^{-1}\mathbf{b}^{(2)}(\mathbf{X},\tau)\, e^{i\theta(\mathbf{x},t)} + \ldots\} + \text{c.c.}, \tag{2.19b}$$

where c.c. denotes 'complex conjugate'. The local frequency ω^* and wave vector $\mathbf{k}^*$ are defined by

$$\omega^*/\Omega^* = \omega(\mathbf{X},\tau) = -\partial\theta/\partial t, \quad l\mathbf{k}^* = \mathbf{k}(\mathbf{X},\tau) = \nabla\theta, \tag{2.20}$$

so that to lowest order (2.8) leads to the algebraic equations

$$-i\omega\mathbf{u}^{(0)} + 2\mathbf{\Omega}\times\mathbf{u}^{(0)} = -i\mathbf{k}p^{(0)} + i(\mathbf{k}.\mathbf{B})\,\mathbf{b}^{(0)}, \tag{2.21a}$$

$$-i\omega\mathbf{b}^{(0)} = i(\mathbf{k}.\mathbf{B})\,\mathbf{u}^{(0)}, \tag{2.21b}$$

$$\mathbf{k}.\mathbf{u}^{(0)} = \mathbf{k}.\mathbf{b}^{(0)} = 0. \tag{2.21c,d}$$

When the frequency ω is related to the wave vector $\mathbf{k}$ by the dispersion relation $D^s(\mathbf{k},\omega) = 0$, the equations have the four solutions

$$u_i^{(0)} = M_{ij}^s q_j^{(0)sn}, \quad \mathbf{b}^{(0)} = \frac{\mathbf{k}.\mathbf{B}}{s\omega_n}\mathbf{u}^{(0)}, \quad p^{(0)} = \frac{2s}{k}\mathbf{\Omega}.\mathbf{u}^{(0)} \quad (s = \pm 1, n = \pm 1), \tag{2.22}$$

where
$$M_{ij}^s(\mathbf{k}) = \tfrac{1}{2}\{(\delta_{ij} - \hat{k}_i\hat{k}_j) + is\epsilon_{inj}\hat{k}_n\} \tag{2.23}$$

may be naturally termed the 'helicity projection operator', the $\mathbf{q}^{(0)sn}$ are arbitrary vectors and the summation convection is restricted to repeated subscripts. Since M_{ij}^s defines the shape of the energy spectrum tensor for random waves [see (3.9) below] it plays a central role in the subsequent theory. Indeed M_{ij}^s performs an almost identical function to the projection operator $\delta_{ij} - \hat{k}_i\hat{k}_j$ used extensively in turbulence theories for incompressible flows. The introduction of M_{ij}^s at this early stage has certain attractions. First, it defines forcefully the mathematical structure of the perturbations, and second, it reduces algebraic manipulation to a minimum. Several useful identities which demonstrate the principal properties of M_{ij}^s are listed in appendix A. In particular its relation to helicity hinges on the fact that, except for a factor k, taking is curl leaves it unaltered or changes its sign depending on the sign of s:

$$i\epsilon_{ijk}k_j M_{kl}^s = skM_{il}^s. \tag{2.24}$$

The helicity property of the waves (2.22) has been considered in detail in I.

Determination of the higher-order terms depends on solving successively the system of algebraic equations

$$-i\omega\mathbf{u}+2\mathbf{\Omega}\times\mathbf{u}+i\mathbf{k}p-i(\mathbf{k}.\mathbf{B})\,\mathbf{b}=\mathscr{F}, \tag{2.25a}$$

$$-i\omega\mathbf{b}-i(\mathbf{k}.\mathbf{B})\,u=\mathscr{G}, \tag{2.25b}$$

$$i\mathbf{k}.\mathbf{u}=a, \quad i\mathbf{k}.\mathbf{b}=c. \tag{2.25c, d}$$

Thus, for example, the order-Q^{-1} terms $\mathbf{u}^{(2)}$ and $\mathbf{b}^{(2)}$ in the expansions of $\mathbf{u}$ and $\mathbf{b}$ are determined when $\mathscr{F}$, $\mathscr{G}$, a and c take the values given by (2.29). The formulation here is, however, general and applicable to the nonlinear case discussed in §4. Multiplication of the jth components of (2.25a, b) by $M_{ij}^{-s'}$ leads to the simpler set of equations

$$[-i\omega\mathbf{u}^{s'}+2is'(\hat{\mathbf{k}}.\mathbf{\Omega})\,\mathbf{u}^{s'}-i(\mathbf{k}.\mathbf{B})\,\mathbf{b}^{s'}]_i=M_{ij}^{-s'}\left[\mathscr{F}_j+\frac{2as'}{k}\,\Omega_j\right], \tag{2.26a}$$

$$[-i\omega\mathbf{b}^{s'}-i(\mathbf{k}.\mathbf{B})\,\mathbf{u}^{s'}]_i=M_{ij}^{-s'}\mathscr{G}_j, \tag{2.26b}$$

where
$$u_i^{s'}=M_{ij}^{-s'}u_j, \quad \mathbf{b}^{s'}=M_{ij}^{-s'}b_j. \tag{2.27a}$$

The total velocity can now be derived from the trivial identities $u_i=\delta_{ij}u_j$ and (A 1) (see appendix A). It is

$$\mathbf{u}=\mathbf{u}^1+\mathbf{u}^{-1}-iak^{-2}\mathbf{k}. \tag{2.27b}$$

Provided that the frequency ω and wave vector $\mathbf{k}$ imposed by the inhomogeneous forcing term do not satisfy the dispersion relation $D^{s'}(\mathbf{k},\omega)=0$, equations (2.26) may be solved, giving

$$u_i^{s'}=\frac{-iM_{ij}^{-s'}}{D^{s'}(\mathbf{k},\omega)}\left\{\omega\mathscr{F}_j+s'\,\frac{2a\omega}{k}\,\Omega_j-(\mathbf{k}.\mathbf{B})\,\mathscr{G}_j\right\}, \tag{2.28a}$$

$$b_i^{s'}=\frac{iM_{ij}^{-s'}}{D^{s'}(\mathbf{k},\omega)}\left\{-(\mathbf{k}.\mathbf{B})\,\mathscr{F}_j-s'\,\frac{2a}{k}(\mathbf{k}.\mathbf{B})\,\Omega_j+(\omega-2s'(\hat{\mathbf{k}}.\mathbf{\Omega}))\,\mathscr{G}_j\right\}. \tag{2.28b}$$

Within the framework of the linearized equations (2.8) the first corrections to the amplitudes of the waves, namely $\mathbf{u}^{(2)}$ and $\mathbf{b}^{(2)}$, stem from the gradient and time derivative of $\mathbf{u}^{(0)sn}$ and $\mathbf{B}$ occurring over the long length and time scales together with the diffusion term $-Q^{-1}k^2\mathbf{b}^{(0)sn}$. Thus the inhomogeneous forcing terms on the right sides of (2.25) become

$$\mathscr{F}^{(0)}=\frac{\partial\mathbf{u}^{(0)}}{\partial\tau}-\nabla p^{(0)}+(\mathbf{B}.\nabla)\,\mathbf{b}^{(0)}+(\mathbf{b}^{(0)}\nabla)\,\mathbf{B}, \tag{2.29a}$$

$$\mathscr{G}^{(0)}=-\partial\mathbf{b}^{(0)}/\partial\tau-(\mathbf{u}^{(0)}.\nabla)\,\mathbf{B}+(\mathbf{B}.\nabla)\,\mathbf{u}^{(0)}-k^2\mathbf{b}^{(0)}, \tag{2.29b}$$

$$a^{(0)}=-\nabla.\mathbf{u}^{(0)}, \quad c^{(0)}=-\nabla.\mathbf{b}^{(0)}. \tag{2.29c, d}$$

Since $D^{-s}(\mathbf{k},\omega)\neq 0$, the contributions $\mathbf{u}^{-s}$ and $\mathbf{b}^{-s}$ to (2.27) are readily determined from (2.28), but because $D^s(\mathbf{k},\omega)=0$ solutions for $\mathbf{u}^s$ and $\mathbf{b}^s$ are only possible when

$$M_{ij}^{-s}\left\{\mathscr{F}_j^{(0)}+2\,\frac{sa^{(0)}}{k}\,\Omega_j-\frac{\mathbf{k}.\mathbf{B}}{s\omega_n}\,\mathscr{G}_j\right\}=0 \tag{2.30a}$$

[see (2.28a)]. Equation (2.30a), which provides a consistency condition on the expansion procedure, may be expressed on multiplication by $q_i^{(0)sn*}$ in the alternative form

$$\mathbf{u}^{(0)*}\cdot\mathscr{F}^{(0)}+p^{(0)*}a^{(0)}+\mathbf{b}^{(0)*}\cdot\mathscr{G}^{(0)}=0. \tag{2.30b}$$

After some lengthy but routine algebraic reductions, which are perhaps best accomplished using suffix notation aided by the important identity

$$u_j^{(0)*}u_k^{(0)}=q_i^{(0)sn*}M_{ij}^{-s}M_{kl}^{-s}q_l^{(0)sn}=\phi^{(0)sn}M_{jk}^{s}, \tag{2.31a}$$

where

$$\mathbf{u}^{(0)*}\cdot\mathbf{u}^{(0)}=q_i^{(0)sn*}M_{ij}^{-s}q_j^{(0)sn}=\phi^{(0)sn} \tag{2.31b}$$

(see appendix A), equation (2.30b) leads to the transport equation

$$\partial E^{(0)}/\partial\tau+\nabla\cdot(\mathbf{c}_g E^{(0)})=-2\omega_i E^{(0)}, \tag{2.32}$$†

modified by magnetic diffusion. Here $E^{(0)}$ is the total wave energy:

$$E^{(0)}=|\mathbf{u}^{(0)}|^2+|\mathbf{b}^{(0)}|^2=\{1+(\mathbf{k}\cdot\mathbf{B}/\omega_n)^2\}\phi^{(0)sn}, \tag{2.33}$$

while the group velocity $\mathbf{c}_g$ and decay rate ω_i are defined by (2.17) and (2.18) respectively. Equation (2.32) shows that wave energy is transported at the group velocity except for that lost by ohmic dissipation.

Without the decay term in (2.32) the system becomes conservative and the result given by Acheson (1972, equation (4.23)) is recovered. The resulting conservation equation is typical of a wide class of problems encountered in the literature. It is perhaps worth noting that action E/ω rather than energy is often the conserved quantity. Such equations are usually derived by the Lagrangian approach of Whitham (1965). Unfortunately this technique appears unsuited to non-conservative systems which result in the presence of dissipation. Consequently equations of the type (2.32) are usually only postulated. The additional decay term, however, introduces no new difficulties for the method of multiple length scales adopted here.

Of particular relevance to the dynamo problem is the mean electromotive force

$$\mathscr{E}=\mathscr{E}^{(0)}+Q^{-1}\mathscr{E}^{(2)}+\ldots=\langle\mathbf{u}\times\mathbf{b}\rangle, \tag{2.34}$$

which provides the source of mean magnetic field. It is immediately apparent from (2.22) that $\mathbf{u}^{(0)*}\times\mathbf{b}^{(0)}$ is pure imaginary and consequently $\mathscr{E}^{(0)}$ vanishes. The first non-vanishing contribution to $\mathscr{E}$ is of order Q^{-1} and given by

$$\mathscr{E}^{(2)}=\mathbf{u}^{(0)*}\times\mathbf{b}^{(2)}+\mathbf{u}^{(2)}\times\mathbf{b}^{(0)*}+\text{c.c.} \tag{2.35}$$

Here $\mathbf{u}^{(2)}$ and $\mathbf{b}^{(2)}$ are derived from (2.26) and (2.27), where $\mathscr{F}$ and $\mathscr{G}$ are defined by (2.29). The terms may be reduced considerably with the aid of identity (A 4) (see appendix A) to give

$$\mathbf{u}^{(0)*}\times\mathbf{b}^{(2)}=(-is/k)\{(\mathbf{u}^{(0)*}\cdot\mathbf{b}^{(2)})\,\mathbf{k}-\mathbf{u}^{(0)*}(\mathbf{k}\cdot\mathbf{b}^{(2)})\}, \tag{2.36a}$$

$$\mathbf{u}^{(2)}\times\mathbf{b}^{(0)*}=(is/k)\{(\mathbf{u}^{(2)}\cdot\mathbf{b}^{(0)*})\,\mathbf{k}-\mathbf{b}^{(0)*}(\mathbf{k}\cdot\mathbf{u}^{(2)})\}. \tag{2.36b}$$

† Henceforth the ith component of the gradient operator ∇ is $\partial/\partial X_i$, except in (3.3b) and (4.9) below.

Use of (2.26)–(2.29) then yields

$$\mathscr{E}^{(2)} = -\frac{1}{\omega_n}\mathbf{u}^{(0)*}.\left\{\frac{\partial \mathbf{b}^{(0)}}{\partial \tau} + (\mathbf{u}^{(0)}.\nabla)\mathbf{B} - (\mathbf{B}.\nabla)\mathbf{u}^{(0)} + k^2\mathbf{b}^{(0)}\right\}\hat{\mathbf{k}} + \frac{1}{k}\mathbf{u}^{(0)*}\mathbf{u}^{(0)}.\nabla\left(\frac{\mathbf{k}.\mathbf{B}}{\omega_n}\right), \tag{2.37a}$$

or

$$\mathscr{E}^{(2)} = \frac{1}{\omega_n}\left\{\frac{s\omega_n}{\mathbf{k}.\mathbf{B}}\frac{\partial}{\partial \tau}\left[\left(\frac{\mathbf{k}.\mathbf{B}}{s\omega_n}\right)^2 \phi^{(0)sn}\right] + \phi^{(0)sn}\hat{k}_i\hat{k}_j\frac{\partial B_i}{\partial X_j} + \mathbf{B}.\nabla\phi^{(0)sn}\right\}\hat{\mathbf{k}}$$
$$-\frac{s}{k}\phi^{(0)sn}\hat{\mathbf{k}}\times\left[\hat{\mathbf{k}}\times\nabla\left(\frac{\mathbf{k}.\mathbf{B}}{s\omega_n}\right)\right] + 2\frac{k^2}{\omega_n}\left(\frac{\mathbf{k}.\mathbf{B}}{s\omega_n}\right)\phi^{(0)sn}\hat{\mathbf{k}}. \tag{2.37b}$$

3. Random waves

The analysis of §2 for a single wave train is now extended to the case of a random distribution of waves. This aspect of the problem is likely to be of greater interest in geophysical and astrophysical applications as it is difficult to envisage a single wave train providing the mechanism for dynamo maintenance! An assumption in turbulence theory that introduces considerable simplifications is that of spatial homogeneity. It was pointed out in I, however, that such an assumption is untenable for the model considered here in §5. The difficulty can be avoided by postulating local homogeneity. In other words the turbulence is assumed to be homogeneous on the short length scale l but not on the long length scale L. Indeed whenever the background medium varies over a long length scale the stronger assumption of global homogeneity cannot be valid. Now for the purpose of solving the dynamo problem it will transpire that it is sufficient to determine the energy spectrum [cf. (2.37*b*) and (4.1) below]; a quantity whose evolution is governed by an equation with some similarity (but *not* identical) to (2.32) [see (3.13)]. It should be emphasized again that the full generality of the theme developed in this section is not needed for the explicit model considered in §§5–7. Therefore, the reader may, if he wishes, move directly to §4 without losing the continuity of the argument.

It is supposed that initially, at $t = 0$, a particular realization of the flow may be defined by

$$\mathbf{u} = \sum_{s=\pm1}\sum_{n=\pm1}\int \hat{\mathbf{u}}^{(0)sn}(\mathbf{k}_0;\mathbf{X},0)\exp(i\mathbf{k}_0.\mathbf{x})\,d^3k_0, \tag{3.1}$$

where the condition $\hat{\mathbf{u}}^{(0)sn}(\mathbf{k};\mathbf{X},\tau) = [\hat{\mathbf{u}}^{(0)s-n}(-\mathbf{k};\mathbf{X},\tau)]^*$ ensures that $\mathbf{u}$ is real. The decomposition into four distinct modes anticipates the character of the subsequent wave motion. If $\hat{\mathbf{u}}^{(0)sn}$ is independent of $\mathbf{X}$, (3.1) is just the usual Fourier representation. The explicit dependence on $\mathbf{X}$ emphasizes the role of (3.1), namely, that it provides a local Fourier decomposition of the velocity. The initial motion can be regarded as the superposition of infinitesimal wave trains

$$\hat{\mathbf{u}}^{(0)sn}(\mathbf{k}_0;\mathbf{X},0)\exp(i\mathbf{k}_0.\mathbf{x})\,d^3k_0 \tag{3.2}$$

which evolve in the form

$$\hat{\mathbf{u}}^{(0)sn}(\mathbf{k};\mathbf{X},\tau)\exp[i\theta^{sn}(\mathbf{k};\mathbf{x},t)]\,d^3k_0, \tag{3.3a}$$

where

$$\mathbf{k} = \mathbf{k}(\mathbf{k}_0;\mathbf{X},\tau) = \nabla\theta, \tag{3.3b}$$

and the infinitesimal d^3k_0 is constant. For linear theory, at least, $\hat{\mathbf{u}}^{(0)sn}$ is determined by the analysis of § 2, so that the actual motion obtained by superposing the wave trains is

$$\mathbf{u} = \sum_{s=\pm1}\sum_{n=\pm1}\int \hat{\mathbf{u}}^{(0)sn}(\mathbf{k};\mathbf{X},\tau)\exp[i\theta^{sn}(\mathbf{k};\mathbf{x},t)]\,d^3k_0. \tag{3.4}$$

Provided that the mapping $\mathbf{k}_0 \to \mathbf{k}$ for given $(\mathbf{X},\tau)$ is one–one and continuous the curious representation (3.4) is unambiguous. It does, however, highlight the main difficulty, which is that, locally, the exponential behaviour is $e^{i\mathbf{k}.\mathbf{x}}$ while integration is over $\mathbf{k}_0$! Evidently the local Fourier representation must involve a weighting factor

$$[\sigma^{sn}(\mathbf{k};\mathbf{X},\tau)]^{-1} = \partial(k_1,k_2,k_3)/\partial(k_{01},k_{02},k_{03}), \tag{3.5a}$$

so that the velocity (3.4) is written better as

$$\mathbf{u} = \sum_{s=\pm1}\sum_{n=\pm1}\int \sigma^{sn}(\mathbf{k};\mathbf{X},\tau)\,\mathbf{u}^{(0)sn}(\mathbf{k};\mathbf{X},\tau)\exp[i\theta^{sn}(\mathbf{k};\mathbf{x},t)]\,d^3k. \tag{3.5b}$$

For the statistical problem it is supposed that the wave trains are initially randomly distributed and spatially homogeneous on the length scale l.† The velocity correlation at two points separated by the space vector $\mathbf{r}$ (which is order 1) is

$$\langle u_i(\mathbf{x},t)\,u_j(\mathbf{x}+\mathbf{r},t)\rangle = \sum_{\substack{s=\pm1\\ s'=\pm1}}\sum_{\substack{n=\pm1\\ n'=\pm1}}\int \Phi_{ij}^{(0)sn,s'-n'}(\mathbf{k};\mathbf{X},\tau)\,e^{i\mathbf{k}.\mathbf{r}} \times\exp[-i(s'\omega_{n'}-s\omega_n)]\,t\,d^3k_0, \tag{3.6a}$$

where $\omega_n(\mathbf{k}) = -\omega_{-n}(-\mathbf{k})$ and

$$\langle \hat{u}_i^{(0)s'n'}(\mathbf{k}';\mathbf{X},\tau)\,u_j^{(0)sn}(\mathbf{k};\mathbf{X},\tau)\rangle = \delta(\mathbf{k}_0'+\mathbf{k}_0)\,\Phi_{ij}^{(0)sn,s'n'}(\mathbf{k};\mathbf{X},\tau). \tag{3.6b}$$

Here the δ-function dependence on $\mathbf{k}_0$ rather than $\mathbf{k}$ presupposes that the correct integration is in $\mathbf{k}_0$ space, in view of the prescription of the initial data [cf. (3.1)–(3.5)]. Of particular interest is the case

$$s' = s, \quad n' = -n, \tag{3.7}$$

for which the velocity correlation is steady on the short time scale (as opposed to periodic). For this case $\delta(\mathbf{k}'+\mathbf{k}_0) = \sigma^{sn}(\mathbf{k};\mathbf{X},\tau)\,\delta(\mathbf{k}'+\mathbf{k})$ and the mean part of (3.6a) (i.e. the part averaged over the short time scale) is

$$\langle\langle u_i(\mathbf{x},t)\,u_j(\mathbf{x}+\mathbf{r},t)\rangle\rangle = \sum_{s=\pm1}\sum_{n=\pm1}\int \sigma^{sn}(\mathbf{k};\mathbf{X},\tau)\,\Phi_{ij}^{(0)sn,sn}(\mathbf{k};\mathbf{X},\tau)\,e^{i\mathbf{k}.\mathbf{r}}\,d^3k. \tag{3.8}$$

At this point, the linear analysis of § 2 may be taken over *en masse*, except for a factor $\frac{1}{2}$ in (2.33), (2.35) and (2.37), provided that $\phi^{(0)sn}$ is given the new interpretation defined by

$$\Phi_{ij}^{(0)sn,sn} = M_{ij}^{s}\,\phi^{(0)sn}, \tag{3.9a‡}$$

where

$$\Phi_{ii}^{(0)sn,sn} = \phi^{(0)sn}. \tag{3.9b}$$

† As usual, the Fourier transforms $\hat{\mathbf{u}}^{(0)sn}(\mathbf{k};\mathbf{X},\tau)$ can be generalized functions.

‡ The shape of the spectrum tensor is identical to that derived in I on the restrictive assumption of isotropy. Clearly this assumption only influences the scalar $\phi^{(0)sn}$.

It follows that the mean total energy density in real space at $(\mathbf{X}, \tau)$ is

$$\sum_{s=\pm 1} \sum_{n=\pm 1} \int \sigma^{sn}(\mathbf{k}; \mathbf{X}, \tau)\, E^{(0)sn}(\mathbf{k}; \mathbf{X}, \tau)\, d^3k, \tag{3.10a}$$

where
$$E^{(0)} = \tfrac{1}{2}\{|\mathbf{u}^{(0)}|^2 + |\mathbf{b}^{(0)}|^2\}, \tag{3.10b}$$

while the mean value of the generation term $\mathscr{E}^{(2)sn}$ in the magnetic induction equation (4.1) below becomes

$$\sum_{s=\pm 1} \sum_{n=\pm 1} \int \sigma^{sn}(\mathbf{k}; \mathbf{X}, \tau)\, \mathscr{E}^{(2)sn}(\mathbf{k}; \mathbf{X}, \tau)\, d^3k, \tag{3.11a}$$

where
$$\mathscr{E}^{(2)} = \tfrac{1}{2}\{\mathbf{u}^{(0)*} \times \mathbf{b}^{(2)} + \mathbf{u}^{(2)} \times \mathbf{b}^{(0)*} + \text{c.c.}\}. \tag{3.11b}$$

The time-averaged expressions (3.10) and (3.11) suggest that an equation governing the energy spectrum $e^{(0)sn} = \sigma^{sn} E^{(0)sn}$ is more appropriate than (2.32), which governs $E^{(0)sn}$. Such an equation is readily obtained by combining (2.32) and the equation

$$\partial\sigma/\partial\tau + \mathbf{c}_g . \nabla\sigma = \sigma\nabla . \mathbf{c}_g, \tag{3.12}$$

derived in appendix B. The resulting equation,

$$\partial e^{(0)}/\partial\tau + \mathbf{c}_g . \nabla e^{(0)} = -2\omega_i e^{(0)} \tag{3.13}$$

[cf. (2.32)], shows that the energy spectrum $e^{(0)sn}$ is constant at points moving at the group velocity $s\mathbf{c}_g^n$ except for the energy lost by ohmic dissipation. It must be emphasized that all differentiations in (3.13) keep $\mathbf{k}_0$ fixed, so that if an equation for fixed $\mathbf{k}$ is required the operator $\partial/\partial\tau + \mathbf{c}_g . \nabla$ must be replaced by

$$\left[\frac{\partial}{\partial\tau} + \mathbf{c}_g . \nabla\right]_{\mathbf{k}\,\text{fixed}} - \left[\frac{\partial\omega}{\partial X_i}\right]_{\mathbf{k}\,\text{fixed}} \frac{\partial}{\partial k_i} \tag{3.14}$$

(see appendix B). In other words an additional term must be added to the right side of (3.13) to account for energy flow in wave vector space caused by spatial gradients of the frequency.

The key roles of the integrals (3.10) and (3.11) might have been anticipated without recourse to the two-point correlations. For whereas the wave energy $E^{(0)}$ refers to a single entity, namely the wave train, the wave energy density $e^{(0)}$ refers to a unit volume in $\mathbf{k}$ space. It follows that both σ and $e^{(0)}$ can loosely be interpreted as densities; though it is curious to find σ^{-1} rather than σ satisfying the continuity equation (B 4). This curiosity can perhaps be attributed to the fact that density in $\mathbf{k}$ space has the same dimensions as specific volume in real space.

4. Nonlinear wave interactions

The mean magnetic field $\mathbf{B}$ is governed by the averaged magnetic induction equation

$$Q^2\, \partial\mathbf{B}/\partial\tau = \epsilon^2\nabla \times (Q\mathscr{E}) + \nabla^2\mathbf{B}, \tag{4.1}$$

where
$$\epsilon = R_0 Q^{\frac{1}{2}}. \tag{4.2}$$

Provided that neglect of nonlinear wave interactions can be justified the mean electromotive force $\mathscr{E}$ is given by (2.34) for a wave train or (3.11) for random waves.

Unfortunately its justification in §2 was based on the crude limiting procedure Q large but fixed, $R_0 \to 0$. In this limit all nonlinear effects, including here the term $\epsilon^2(Q\mathscr{E})$ in (4.1) ($\epsilon \to 0$), are lost indiscriminately and so dynamo action is impossible. Equations (4.1) and (4.2) do suggest, however, the more discerning limiting procedure ϵ fixed, $R_0 \to 0$. For it can then be argued that, if the results derived so far remain valid, $Q\mathscr{E}$ will stay of order 1 [see (2.34) and (2.35)] and consequently magnetic induction and magnetic diffusion may become comparable.

The principal difficulty associated with the new limit is immediately apparent, namely the relatively large values of the nonlinear terms which occur in the equations governing $\mathbf{u}$ and $\mathbf{b}$ [see (4.9) below]. Despite the fact that the nonlinear terms omitted in (2.8) are only of order R_0, or equivalently order $Q^{-\frac{1}{2}}$, they are an order of magnitude larger than the order-Q^{-1} diffusion term which is retained. Therefore to recover the previous results and extend the theory new expansions such as

$$\mathbf{u} = \mathbf{u}_0 + Q^{-\frac{1}{2}}\mathbf{u}_1 + Q^{-1}\mathbf{u}_2 + \dots \tag{4.3}$$

are considered which use $Q^{-\frac{1}{2}}$ rather than Q^{-1} as the small expansion parameter.

Restricting attention for the moment to the wave train considered in §2, the nonlinear interactions excite a mode at the $Q^{-\frac{1}{2}}$ level proportional to $e^{2i\theta}$. However, no mean flow is generated because the real parts of $(\nabla \times \mathbf{u}^{(0)*}) \times \mathbf{u}^{(0)}$ and $(\nabla \times \mathbf{b}^{(0)*}) \times \mathbf{b}^{(0)}$, which give rise to the mean Reynolds stress in the averaged equation of motion, both vanish. It is assumed that the mean state is static and that the Lorentz force $(\nabla \times \mathbf{B}) \times \mathbf{B}$ is in equilibrium with the pressure force. A further nonlinear interaction between the forced mode $\mathbf{u}_1$ and the free wave $\mathbf{u}_0$ has two effects. It forces an order-Q^{-1} mode proportional to $e^{3i\theta}$ but, more significantly, causes the original free wave $\mathbf{u}_0 = \mathbf{u}^{(0)}e^{i\theta} + \text{c.c.}$ to resonate. The resulting secular behaviour of $\mathbf{u}_2$ is removed, as in the linear case, by (2.30), where now (2.29) contains nonlinear terms. The ultimate effect of the wave–wave interactions is just to introduce a term proportional to $E^{(0)2}$ in (2.32), which accounts for the energy transfer between the fundamental mode and the higher harmonics; transfer of energy to the mean flow is negligible. For the random waves considered in §3 it can also be argued, following Benney & Saffman (1966) and Benney & Newell (1969), that the effects of wave–wave interactions enter the problem at precisely the same level. The arguments are more subtle and depend on the existence of resonant-triad interactions.

In the magnetic induction equation (4.1) the mean electromotive force $\mathscr{E}$ remains of order Q^{-1}. Indeed nonlinearities affect $\mathscr{E}$ at precisely the same level as in the previous linear theory. Consequently the first non-vanishing term in the modified expansion (2.34) is

$$\mathscr{E}^{(2)} = \langle \mathbf{u}_0 \times \mathbf{b}_2 + \mathbf{u}_1 \times \mathbf{b}_1 + \mathbf{u}_2 \times \mathbf{b}_0 \rangle. \tag{4.4}$$

This may be expressed in part by the terms in (2.37*b*), which are all linear in $\phi^{(0)sn}$, together with new terms proportional to $(\phi^{(0)sn})^2$. It follows that the dynamo equation (4.1) and the extended wave energy equation (2.32) provide a coupled pair of equations which determines the evolution of a hydromagnetic dynamo. It should be noted, however, that the presence of the term $Q^2\,\partial\mathbf{B}/\partial\tau$ indicates that the magnetic field evolves over a much longer time scale than the wave energy. Consequently within the framework of the approximations made **B** may be regarded as steady in the wave energy equation (2.32).

Both the wave energy and dynamo equations still contain a dimensionless parameter, namely ϵ. Consequently yet more approximations can be made if ϵ is assumed small, though not so small as to violate any previous approximations based on the smallness of $Q^{-\frac{1}{2}}$. Attention is, therefore, focused on the restricted parameter range

$$1 \gg \epsilon \gg Q^{-\frac{1}{2}}. \tag{4.5}$$

If, for the moment, it is assumed that the linearization procedures are justified, only the terms linear in the wave energy $\phi^{(0)sn}$, namely those which appear explicitly in (2.32) and (2.37), need to be considered. Thus the expression (2.37*b*) for $\mathscr{E}^{(2)}$ suggests that low frequency modes provide the most significant contribution to dynamo maintenance. It is not unreasonable, therefore, to isolate these modes by assuming that

$$\omega = O(\epsilon). \tag{4.6}$$

In this way $\mathscr{E}^{(2)}$ is dominated by the last term in (2.37*b*), namely

$$2\frac{k^2}{\omega_n}\left(\frac{\mathbf{k}.\mathbf{B}}{s\omega_n}\right)\phi^{(0)sn}\,\hat{\mathbf{k}}, \tag{4.7}$$

and if **B** is order 1, this provides an order 1 contribution to $\epsilon^2\mathscr{E}^{(2)}$ in (4.1). It must be emphasized that (4.7) is a direct consequence of ohmic decay and differs in origin from the remaining terms in (2.37*b*), which result from inhomogeneities in the mean magnetic field **B** and the wave energy $\phi^{(0)sn}$. Since the magnetic induction is linear in **B**, it is necessary to consider the dynamical equations to obtain a realistic estimate of the magnitude of **B**. If it is anticipated that ω_i is order 1 in the range of interest, it is reasonable to suppose also that

$$\mathbf{B} = O(\epsilon) \tag{4.8}$$

[see (2.18)].

In order to assess the importance of wave–wave interactions it is necessary to consider the full nonlinear equations

$$\partial\mathbf{u}/\partial t + 2\mathbf{\Omega}\times\mathbf{u} + \epsilon Q^{-\frac{1}{2}}\{(\mathbf{u}.\nabla)\,\mathbf{u} - \langle(\mathbf{u}.\nabla)\,\mathbf{u}\rangle\}$$
$$= -\nabla p + (\mathbf{B}.\nabla)\,\mathbf{b} + (\mathbf{b}.\nabla)\,\mathbf{B} + \epsilon Q^{-\frac{1}{2}}\{(\mathbf{b}.\nabla)\,\mathbf{b} - \langle(\mathbf{b}.\nabla)\,\mathbf{b}\rangle\}, \tag{4.9a}$$

$$\partial\mathbf{b}/\partial t + (\mathbf{u}.\nabla)\,\mathbf{B} = (\mathbf{B}.\nabla)\,\mathbf{u} + \epsilon Q^{-\frac{1}{2}}\nabla\times\{\mathbf{u}\times\mathbf{b} - \langle\mathbf{u}\times\mathbf{b}\rangle\} + Q^{-1}\nabla^2\mathbf{b} \tag{4.9b}$$

governing the fluctuating velocity **u** and magnetic field **b**. Arguing the case for a wave train, it is clear that $\mathbf{u}_0$ and $\mathbf{b}_0$ are of the same order and may both be assumed order 1. The forced modes $\mathbf{u}_1$ and $\mathbf{b}_1$, which have low frequency 2ω, are determined by (2.28). Here $\mathscr{F}$ and $\mathscr{G}$ arise from the nonlinear terms in (4.9) and are of order ϵ. Moreover ω and $\mathbf{k}.\mathbf{B}$ [see (4.6) and (4.8)] are of order ϵ, while $D^s(2\mathbf{k}, 2\omega)$ is of order ϵ^2. Consequently $\mathbf{u}_1$ and $\mathbf{b}_1$ are of order 1. They therefore only provide an order-ϵ contribution to the wave energy equation (2.32) and may legitimately be neglected. Since the contribution to $\mathbf{u}_1$, $\mathbf{b}_1$, $\mathbf{u}_2$ and $\mathbf{b}_2$ from the wave–wave interactions is at most of order 1, the corresponding contribution to $\mathscr{E}^{(2)}$ is of order 1 also. But, since it is anticipated that **B** is of order ϵ [see

(4.8)] the contribution (4.7) to $\mathscr{E}^{(2)}$ is of order ϵ^{-1} and so remains an order of magnitude, ϵ^{-1}, larger than the contribution from the nonlinear terms. Thus it is legitimate to neglect wave–wave interactions completely in the dynamo equation (4.1) as well as the wave energy equation (2.32). As observed earlier, the conclusions arrived at here may be expected to remain valid for random waves though the details of the argument are substantially different.

The physical significance of the dominant contribution (4.7) to the mean electromotive force $\mathscr{E}$ is obscured somewhat by the large number of terms in the expansions of $\mathbf{u}$ and $\mathbf{b}$ required in the derivation. The vanishing of $\mathscr{E}^{(0)}$, however, emphasizes that to lowest order the frozen-field approximation has been made. The added fact that (4.7) leads to a mean electromotive force which is directly proportional to the magnetic diffusivity suggests that the α-effect is of the Braginskiĭ (1964*a*, *b*) type. Since magnetic field lines in this high conductivity limit are almost material lines, it is natural to adopt the Lagrangian rather than the Eulerian viewpoint. The general Lagrangian formulation developed for the Braginskiĭ dynamo by Soward (1972) in a cylindrical geometry is therefore appropriate. In our problem, however, considerable simplifications ensue because of approximations based on the multiple length scales. Indeed the approximate Lagrangian formulations given elsewhere for waves (e.g. see Bretherton 1970) proceed to the required level of accuracy.

Suppose for the moment that the fluid is perfectly conducting and that the magnetic field in the absence of motion is $\mathbf{B}_L(\mathbf{x})$. Then after fluid particles have been displaced a small distance $Q^{-\frac{1}{2}}\boldsymbol{\xi}(\mathbf{x},t)$ from their initial positions $\mathbf{x}$, the strength of the magnetic field at $\mathbf{x}+Q^{-\frac{1}{2}}\boldsymbol{\xi}$ is

$$\mathbf{B}_L + Q^{-\frac{1}{2}}\mathbf{B}_L.\nabla\boldsymbol{\xi}. \tag{4.10}$$

When considering the high conductivity limit $Q \gg 1$, (4.10) still provides a useful representation of the magnetic field. Indeed for the periodic disturbances considered here the Lagrangian magnetic field $\mathbf{B}_L$, correct to order R_0, is just $\mathbf{B}$, in direct contrast to the Eulerian representation (2.5). The equation governing the Lagrangian average $\langle \mathbf{B}_L \rangle$ of the magnetic field is given by Soward (1972, equation (3.3)). After the obvious modifications and approximations based on the multiple length scales have been made, (4.1) is recovered, where now

$$\epsilon^2 Q\mathscr{E}_i = \left\langle \frac{\partial\boldsymbol{\xi}}{\partial x_i}.\nabla\times\frac{\partial\boldsymbol{\xi}}{\partial x_j}\right\rangle B_j. \tag{4.11}$$

Upon approximating the velocity $\mathbf{u}$ at $\mathbf{x}$ by $\epsilon^{-1}\,\partial\boldsymbol{\xi}/\partial t$, the identity (4.11) leads quite simply to (4.7) above.

The simplicity of the α-effect is now readily appreciated in terms of the geometric distortion of a magnetic field line caused by a single wave $\mathrm{Re}\,\hat{\mathbf{u}}(\mathbf{k})\,e^{i(\mathbf{k}.\mathbf{x}-\omega t)}$. Since fluid particles describe circles of small radius, of order $Q^{-\frac{1}{2}}l$, in planes perpendicular to $\mathbf{k}$, the perturbed magnetic field lines have a slightly helical structure (see figure 1). The weak mean electric currents which flow in the direction antiparallel to $\mathbf{k}$, if $s = +1$, are associated solely with magnetic-field distortions and are ultimately responsible for the α-effect.

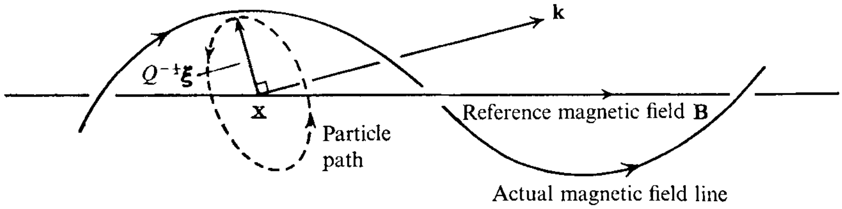

FIGURE 1. The actual magnetic field strength at $\mathbf{x}+Q^{-\frac{1}{2}}\boldsymbol{\xi}$ is $\mathbf{B}+Q^{-\frac{1}{2}}\mathbf{B}.\nabla\boldsymbol{\xi}$, where the ratio of the displacement distance to the wavelength is of small order $Q^{-\frac{1}{2}}$.

5. A dynamo model

The dynamo model described here illustrates the ideas developed in the previous sections, namely the propagation and decay of random waves together with possible concomitant maintenance of the magnetic field. The model is developed within the framework of first-order smoothing, which is justified in the parameter range (4.5), provided that only low frequency modes, of order $\epsilon\Omega^*$, are excited.

The main features of the model have already been introduced in §1. In particular fluid is confined between perfectly conducting planes located at

$$(Q^{-1}z =)\quad Z = 0 \quad \text{and} \quad Z = L_0/L.$$

The fluid is permeated by a large-scale horizontal magnetic field

$$\mathbf{B} = \epsilon\mathbf{B}_H, \tag{5.1}$$

which varies with Z but does not depend on the horizontal co-ordinate. The relatively small strength of the magnetic field $\mathbf{B}$ is suggested by the estimate (4.8), while variations in $\mathbf{B}$ can only occur on the very slow time scale Q^3/Ω^*, as indicated by the magnetic induction equation (4.1). Waves having fixed horizontal wavenumber $|\mathbf{k}_H| = 1$, given low frequency $\epsilon\omega_0$ ($=\omega_0^*/\Omega^*$), but random orientation

$$\mathbf{k}_H(\chi) = (\cos\chi, \sin\chi) \tag{5.2}$$

are excited homogeneously at $Z = 0$. Since the decay time Q/Ω^* for the waves is short compared with the dynamo time scale, the magnetic field may be regarded as steady and a function of Z only in the dispersion relation (2.10). Consequently the waves propagate up through the fluid with both their frequency and horizontal wave vector fixed. The small strength $\epsilon\mathbf{B}_H$ of the magnetic field, together with the low frequency $\epsilon\omega_0$, ensures that the vertical wavenumber

$$k_z = \epsilon[\omega_0^2 - (\mathbf{k}_H.\mathbf{B}_H)^2]/2s\omega_0 + O(\epsilon^2) \quad (s = \pm 1) \tag{5.3}$$

is small.† It follows that the phase velocity of the waves is almost horizontal, while the group velocity

$$\mathbf{c}_g = \frac{2s}{k}\left\{1 + \left(\frac{\mathbf{k}_H.\mathbf{B}_H}{\omega_0}\right)^2\right\}^{-1}\boldsymbol{\Omega} + O(\epsilon) \tag{5.4}$$

† The two limits $\omega_0 \gg \mathbf{k}_H.\mathbf{B}$ and $\omega_0 \ll \mathbf{k}_H.\mathbf{B}$ correspond respectively to the inertial and magnetic waves investigated by Braginskiĭ (1967) using the WKB method.

is almost parallel to the rotation axis. The waves, of course, suffer some attenuation due to ohmic dissipation, as indicated by (2.32), and are ultimately absorbed perfectly at $Z = L_0/L$. Since wave energy is transported upwards, only waves for which $\mathbf{c}_g.\mathbf{\Omega} > 0$ are possible. Consequently (5.4) indicates that only the single wave defined by

$$s = +1 \tag{5.5}$$

in (5.3) prevails.

It is clear that random waves of the generality discussed in §3 are not envisaged here. Indeed a particular realization of the flow may be defined by

$$\mathbf{u}_0 = \int_0^{2\pi} \hat{\mathbf{u}}^{(0)}(\chi; Z) \exp\left[i\left(\mathbf{k}_H(\chi).\mathbf{x} + \int_0^z k_z dz - \epsilon\omega_0 t\right)\right] d\chi + \text{c.c.}, \tag{5.6}$$

which is the sum of two single integrals over the angle χ rather than triple integrals throughout Fourier space. The new velocity correlation corresponding to (3.6) then takes the form

$$\langle u_{0i}(\mathbf{x},t)\, u_{0j}(\mathbf{x}+\mathbf{r},t)\rangle = 2\int_0^{2\pi} \Phi_{ij}^{(0)}(\chi; Z)\, e^{i\mathbf{k}(\chi).\mathbf{r}}\, d\chi, \tag{5.7a}$$

where $$\langle \hat{u}_i^{(0)*}(\chi'; Z)\, \hat{u}_j^{(0)}(\chi; Z)\rangle = \delta(\chi - \chi')\, \Phi_{ij}^{(0)}(\chi; Z). \tag{5.7b}$$

As in §3, $\Phi_{ij}^{(0)}$ is identified with $M_{ij}^1 \phi$ but this time, owing to the presence of the second term in (5.6), summarized by the abbreviation c.c., and the resulting 2 in (5.7*a*), equations (2.33), (2.35) and (2.37) are not modified by the factor $\frac{1}{2}$. Since it is natural to retain the parameter χ and not to introduce the local wave vector, the complications originating from the density σ introduced in §3 are avoided. The simplifications ensuing from (5.7) more than compensate for the loss of generality imposed by the assumed form (5.6). It is reasonable to expect that, though these restrictions affect the quantitative results, the general conclusions of the model are qualitatively correct.

The velocity scale is normalized such that ϕ is unity on the boundary $Z = 0$. Consequently, after all transients have died away, ϕ varies on the very slow time scale Q^3/Ω^*. The value of ϕ is determined by the wave energy equation (2.32), which to lowest order in both Q and ϵ is

$$\partial\phi/\partial\xi = -(\mathbf{k}_H.\mathbf{B}_H)^2\phi \quad (L_M\xi = z^*), \tag{5.8a}$$

where curiously the length

$$L_M = \omega_0^2 L = l^3\omega_0^{*2}/U_0^2 \tag{5.8b}$$

is independent of the magnetic diffusivity λ. The solution satisfying the boundary condition is

$$\phi = \exp\left\{-\int_0^{\xi} (\mathbf{k}_H.\mathbf{B}_H)^2 d\xi\right\}. \tag{5.9}$$

To the same order of approximation $Q\mathscr{E}$ in (4.1) is just $\mathscr{E}^{(2)}$ and given approximately by (4.7). Consequently the dynamo equation (4.1) reduces to

$$\frac{\partial\mathbf{B}_H}{\partial T} = \frac{\partial}{\partial\xi}\left\{2\mathbf{\Omega}\times\int_0^{2\pi} \mathbf{k}_H(\mathbf{k}_H.\mathbf{B})\,\phi(\chi;\xi)\, d\chi\right\} + \frac{\partial^2\mathbf{B}_H}{\partial\xi^2} \quad \left(\frac{L_M^2 T}{\lambda} = t^*\right). \tag{5.10}$$

Since the boundaries are perfectly conducting the horizontal component of the mean electric field vanishes there. Consequently the boundary condition which must be applied to the magnetic field is

$$2\mathbf{\Omega}\times\int_0^{2\pi}\mathbf{k}_H\,(\mathbf{k}_H\,.\,\mathbf{B}_H)\,\phi(\chi;\xi)\,d\chi+\frac{\partial\mathbf{B}_H}{\partial\xi}=0\quad(\xi=0,L_0/L_M). \tag{5.11}$$

This condition ensures that

$$\frac{\partial}{\partial T}\left\{\int_0^{L_0/L_M}\mathbf{B}_H\,d\xi\right\}=0.$$

Hence, if

$$\int_0^{L_0/L_M}\mathbf{B}_H\,d\xi=0 \tag{5.12}$$

initially it remains so. Of course, the statement (5.12) is necessary if the problem considered is to be that of dynamo maintenance! When the magnetic field is steady ($\partial\mathbf{B}_H/\partial T=0$), integration of (5.10) shows that (5.11) holds everywhere in the interval $0\leqslant\xi\leqslant L_0/L_M$ and not just on the boundaries.

The mathematical statement of the hydromagnetic dynamo problem is now complete and specified by the equations (5.9) and (5.10), the boundary condition (5.11) and the initial condition (5.12). The solutions, which are characterized by the single dimensionless parameter

$$\Delta=L_0/L_M=L_0\,U_0^2/l^3\omega_0^{*2}, \tag{5.13}$$

are determined by analytic and numerical methods in the next two sections. The actual partitioning of wave energy between kinetic and magnetic, however, depends on the additional order 1 ratio

$$L/L_M=\omega_0^{-2}=\Omega^*U_0^2/\lambda\omega_0^{*2} \tag{5.14}$$

[see (7.11) below]. For waves of given wavelength and frequency, Δ may be regarded as either a measure of the separation L_0 of the planes or the kinetic energy density $2\pi\rho U_0^2$ of the emitted waves. Whenever dynamo action is possible, the strength of the resulting magnetic field is of prime importance† as it indicates the efficiency of the dynamo mechanism. A convenient measure of the magnetic field strength is provided by the total magnetic energy per unit area in the horizontal plane, namely

$$\mathscr{M}=Q(\rho L_M U_0^2)\,M, \tag{5.15a}$$

where

$$M=\frac{1}{2}\int_0^{\Delta}|\mathbf{B}_H|^2\,d\xi. \tag{5.15b}$$

Except for numerical factors such as Δ, L/L_M and M, the mean magnetic energy is larger than the corresponding kinetic or total wave energy by a factor Q.

6. The weak-field case

The investigation of the weak-field case is begun by making the simplifying assumption that the Lorentz force may be neglected completely. In this case pure inertial waves propagate without attenuation and consequently ϕ takes

† This is in contrast to the case for a kinetic dynamo, where the strength of the magnetic field is determined up to a multiplicative constant.

the value unity everywhere. The magnetic induction term is now readily determined from the identity

$$\int_0^{2\pi} \mathbf{k}_H(\mathbf{k}_H\,.\,\mathbf{B})\,d\chi = \pi\mathbf{B}_H \tag{6.1}$$

and leads to the linear equation

$$\frac{\partial}{\partial T}\begin{bmatrix} B_x \\ B_y \end{bmatrix} = 2\pi\frac{\partial}{\partial \xi}\begin{bmatrix} -B_y \\ B_x \end{bmatrix} + \frac{\partial^2}{\partial \xi^2}\begin{bmatrix} B_x \\ B_y \end{bmatrix}, \tag{6.2}$$

which governs the evolution of the resulting kinematic dynamo. The eigenmode with the fastest growth rate r is

$$B_x = A\,e^{rT}\left\{\cos\frac{\pi\xi}{\Delta}\cos(\pi\xi-\psi) - \Delta\sin\frac{\pi\xi}{\Delta}\sin(\pi\xi-\psi)\right\}, \tag{6.3a}$$

$$B_y = -A\,e^{rT}\left\{\cos\frac{\pi\xi}{\Delta}\sin(\pi\xi-\psi) + \Delta\sin\frac{\pi\xi}{\Delta}\cos(\pi\xi-\psi)\right\},$$

where

$$m/\pi^2 = 1 - 1/\Delta^2, \tag{6.3b}$$

and the amplitude factor A and the phase ψ are arbitrary constants. Evidently dynamo action is not possible when $\Delta < 1$, while the growth rate r is positive when $\Delta > 1$, and increases to its maximum value π^2 as $\Delta \to \infty$. The solution (6.3) represents the sums of two distinct modes [see equation (4.15) in I] of the unbounded problem with the same growth rate. At least one of these modes varies on the length scale L_M. It is a reflexion of the sensitivity of the solution to the boundary conditions that two modes, which are almost coincident with the mode of maximum growth rate, are selected as $L_0 \to \infty$. It may be anticipated (but is justified later in the section) that in the full hydromagnetic dynamo the marginal case $\Delta = 1$ corresponds to zero magnetic field strength: $A = M = 0$. The result is, of course, consistent with the assumptions already made.

It is immediately apparent from (5.13) and (6.3) that, if the intensity of the emitted waves is fixed, dynamo action is always possible provided that the separation L_0 of the planes is sufficiently large. Moreover, it is perhaps significant that the criterion $\Delta \geqslant 1$, necessary for dynamo action, is independent of the angular speed Ω^* and the magnetic diffusivity λ [see (5.13)]. Indeed, the latter result indicates that the amplitude of the α-effect is proportional to λ and suggests that dynamo action is possible in the limit of perfect conductivity $\lambda \to 0$.

When Δ is increased slightly above its critical value unity to the new value

$$\Delta = 1 + \delta \quad (\delta \ll 1), \tag{6.4}$$

a seed magnetic field will begin to grow on the extremely slow time scale $Q^3/\delta\Omega^*$. A small fraction of the wave energy is now dissipated as it is transported across the gap and consequently magnetic induction is rendered less efficient. Ultimately, a finite amplitude steady state is achieved in which magnetic induction exactly compensates for ohmic diffusion. The nonlinear process may be analysed in detail by adopting a perturbation procedure in which (6.3) with Δ set equal to unity provides the lowest-order solution. As a first approximation (6.3) is, however, inadequate because the amplitude A and phase angle ψ are undetermined. The principal objective of the calculation is, therefore, to derive equations (6.11) below, which predict the development of $A = \delta^{\frac{1}{2}}a$ and ψ on the $Q^3/\delta\Omega^*$ time scale to the ultimate steady state.

For convenience the magnetic field is referred to co-ordinate axes rotated through an angle ψ about the ξ axis and written as

$$\mathbf{B}_H = B_{\parallel}(\xi, T_1)\,\mathbf{k}_H(\psi) + B_{\perp}(\xi, T_1)\boldsymbol{\Omega}\times\mathbf{k}_H(\psi) \quad (T_1 = \delta T). \tag{6.5}$$

The angle ψ is chosen to be the phase angle defined by the first approximation (6.3) and, since $\psi(T_1)$ may vary with time, the new reference frame in general rotates. It transpires that $\mathbf{B}_H$ is of order $\delta^{\frac{1}{2}}$ and so the magnetic field and energy spectrum have the expansions

$$\begin{bmatrix} B_{\parallel} \\ B_{\perp} \end{bmatrix} = \delta^{\frac{1}{2}}a(T_1)\begin{bmatrix} \cos 2\pi\xi \\ -\sin 2\pi\xi \end{bmatrix} + \delta^{\frac{3}{2}}\begin{bmatrix} B_{\parallel}^{(1)} \\ B_{\perp}^{(1)} \end{bmatrix} + O(\delta^{\frac{5}{2}}), \tag{6.6a}$$

$$\phi = 1 + \delta\phi^{(1)} + O(\delta^2). \tag{6.6b}$$

The value of $\phi^{(1)}$ determined by expanding (5.9) is

$$\phi^{(1)} = -\tfrac{1}{2}a^2\{\xi + (4\pi)^{-1}[\sin 2(2\pi\xi + \chi - \psi) - \sin 2(\chi - \psi)]\}. \tag{6.7}$$

After substitution of (6.6) and (6.7) into (5.10) the equation

$$\frac{\partial^2}{\partial\xi^2}\begin{bmatrix} B_{\parallel}^{(1)} \\ B_{\perp}^{(1)} \end{bmatrix} + 2\pi\frac{\partial}{\partial\xi}\begin{bmatrix} -B_{\perp}^{(1)} \\ B_{\parallel}^{(1)} \end{bmatrix} = \frac{da}{dT_1}\begin{bmatrix} \cos 2\pi\xi \\ -\sin 2\pi\xi \end{bmatrix} + a\frac{d\psi}{dT_1}\begin{bmatrix} \sin 2\pi\xi \\ \cos 2\pi\xi \end{bmatrix}$$
$$+ a^3\frac{d}{d\xi}\begin{bmatrix} \pi\xi\sin 2\pi\xi \\ \pi\xi\cos 2\pi\xi + \frac{1}{4}\sin 2\pi\xi \end{bmatrix} \tag{6.8}$$

is obtained for the first correction to the magnetic field, namely $\mathbf{B}_H^{(1)}$. A simple integration with respect to ξ and application of the boundary condition (5.11) on $\xi = 0$ is followed by further integration, yielding the solution

$$\begin{bmatrix} B_{\parallel}^{(1)} \\ B_{\perp}^{(1)} \end{bmatrix} = \frac{1}{4\pi^2}\frac{da}{dT_1}\begin{bmatrix} 2\pi\xi\sin 2\pi\xi + (\cos 2\pi\xi - 1) \\ 2\pi\xi\cos 2\pi\xi - \sin 2\pi\xi \end{bmatrix}$$
$$- \frac{a}{4\pi^2}\frac{d\psi}{dT_1}\begin{bmatrix} 2\pi\xi\cos 2\pi\xi - \sin 2\pi\xi \\ -2\pi\xi\sin 2\pi\xi - (\cos 2\pi\xi - 1) \end{bmatrix}$$
$$+ \frac{a^3}{8\pi}\begin{bmatrix} (2\pi\xi)^2\sin 2\pi\xi - \pi\xi\cos 2\pi\xi + \frac{1}{2}\sin 2\pi\xi \\ (2\pi\xi)^2\cos 2\pi\xi + \pi\xi\sin 2\pi\xi \end{bmatrix}. \tag{6.9}$$

The remaining boundary condition on $\xi = \Delta$ is met provided that the initial condition (5.12) is satisfied. After substitution of (6.9) into (6.6), the integral requirement

$$\int_0^{\Delta}\mathbf{B}_H\,d\xi = O(\delta^{\frac{5}{2}}) \tag{6.10}$$

gives the two equations

$$da/dT_1 = \pi^2(2 - \tfrac{1}{2}a^2)\,a, \tag{6.11a}$$

$$d\psi/dT_1 = \tfrac{3}{8}\pi a^2 \tag{6.11b}$$

governing the evolution of a and ψ.

Suppose that initially an arbitrary seed magnetic field $\mathbf{B}_H(\xi, 0)$ is introduced into the system. All the eigenmodes of (6.2) then decay except for (6.3), which has a fixed phase angle ψ and a slow growth rate $r = 2\pi^2\delta$. The finite magnetic field, together with ohmic dissipation, causes the wave energy density to decrease monotonically across the gap. The corresponding reduction of the α-effect becomes significant when the magnetic energy M per unit area [see (5.15)] is of order δ. Mathematically the reduction is manifested by the nonlinear terms of (6.11). In the nonlinear regime the growth rate $r = \pi^2(2-\frac{1}{2}a^2)$ begins to decrease, while the phase angle ψ begins to increase. Both the amplitude a and the phase speed $d\psi/dT$ continue to increase monotonically and approach the steady state, in which $a = 2$ and $d\psi/dT_1 = \frac{3}{2}\pi$, as $T_1 \to \infty$. If, for some reason, a was initially greater than 2, the steady state would be approached from above.

A more careful analysis of the development from a seed magnetic field would take account of possible variations in $\mathbf{B}$ on the $\delta^{-\frac{1}{2}}L$ length scale in the horizontal plane. In this case the additional diffusion terms

$$\nabla_\delta^2 a - (\nabla_\delta \psi)^2 a, \quad a\nabla_\delta^2\psi + 2\nabla_\delta a \,.\, \nabla_\delta \psi \quad (\nabla_\delta \equiv (\delta^{-\frac{1}{2}}\partial/\partial X, \quad \delta^{-\frac{1}{2}}\partial/\partial Y)) \tag{6.12}$$

appear on the right sides of (6.11a, b) respectively. Though solution of the nonlinear equations then becomes difficult, it is a simple matter to show that the finite amplitude solution derived above is stable to arbitrary perturbations.

In the steady state the total magnetic energy per unit area is

$$2\delta Q(\rho L_M U^2), \tag{6.13a}$$

the mean magnetic field has the sheared character described by (6.6a), while relative to the original co-ordinate system, individual magnetic field lines rotate about the z^* axis with angular velocity

$$\tfrac{3}{2}\pi\delta\lambda/L_M^2. \tag{6.13b}$$

The key factor causing the rotation is the spatial attenuation of the waves propagated from $z^* = 0$, which causes regeneration of the magnetic field to be non-uniform. Thus, though the constant-magnitude magnetic field (6.6a) ($\Delta = 1$) was investigated in I, its rotational properties could not be anticipated on the basis of global homogeneity of the initial turbulence.

7. A finite amplitude steady dynamo

When $\Delta - 1$ is no longer small, the perturbation analysis of §6 becomes inadequate. The steady finite amplitude solution described in the last paragraph suggests, however, a form the solution could take in the general case. In particular, a solution is sought in which the magnetic field is steady,

$$\mathbf{B} = (B_\parallel(\xi), B_\perp(\xi)), \tag{7.1}$$

with respect to a frame of reference which rotates with constant angular velocity

$$n = d\psi/dT. \tag{7.2}$$

Such a solution is stable when $\Delta - 1$ is small and there is no reason to suppose that this state of affairs changes outside the region in parameter space for which the perturbation analysis is justified. In this section the steady solution (6.13) is extended by numerical computation to large values of Δ. Similar families of solutions could be obtained by starting with higher-order modes which are initiated when Δ takes integer values greater than 1. It seems likely that for any given Δ steady solutions from the latter classes correspond to smaller magnetic energies. Moreover it may be speculated that such solutions are unstable and that ultimately one of the steady solutions described in this section is attained.

The assumptions (7.1) and (7.2) introduce considerable simplifications into the governing equations (5.8)–(5.12). First, it is easy to show that the energy spectrum (5.9) is

$$\phi = \exp\{-m(\xi)-g(\xi)\cos[2(\chi-nT)-\alpha]\}, \tag{7.3a}$$

where

$$m(\xi) = \frac{1}{2}\int_0^\xi (B_\parallel^2+B_\perp^2)\,d\xi \tag{7.3b}$$

and

$$g(\xi)\,e^{i\alpha(\xi)} = \frac{1}{2}\int_0^\xi (B_\parallel+iB_\perp)^2\,d\xi. \tag{7.3c}$$

Second, (7.3) can be substituted into (5.10), yielding the equation

$$n\begin{bmatrix}-B_\perp\\ B_\parallel\end{bmatrix} = 2\pi\frac{\partial m}{\partial \xi}\left\{e^{-m}\begin{bmatrix}-B_\perp I_0(-g)+(-B_\parallel\sin\alpha+B_\perp\cos\alpha)I_1(-g)\\ B_\parallel I_0(-g)+(B_\parallel\cos\alpha+B_\perp\sin\alpha)I_1(-g)\end{bmatrix}\right\}+\frac{\partial^2}{\partial\xi^2}\begin{bmatrix}B_\parallel\\ B_\perp\end{bmatrix}, \tag{7.4†}$$

where I_0 and I_1 are the zero- and first-order Bessel functions of imaginary argument. The advantages gained by assuming this form of solution are considerable. Not only has the time dependence been eliminated from the problem but also the dependence on the angle χ. The problem has thus been reduced to solving a nonlinear ordinary differential equation with two-point boundary conditions. For given Δ the value of n which gives a solution satisfying the boundary conditions can be thought of as an eigenvalue. The solution $(B_\parallel, B_\perp)$ is then the eigenvector, for which the total magnetic energy M provides a convenient norm.

For computational purposes it is convenient to write (7.3) and (7.4) in the form

$$\partial D_\parallel/\partial\xi = -nB_\perp, \quad \partial D_\perp/\partial\xi = nB_\parallel, \tag{7.5a, b}$$

$$\partial B_\parallel/\partial\xi = D_\parallel + 2\pi e^{-m}\{B_\perp I_0(-g)+(B_\parallel\sin\alpha - B_\perp\cos\alpha)I_1(-g)\}, \tag{7.5c}$$

$$\partial B_\perp/\partial\xi = D_\perp - 2\pi e^{-m}\{B_\parallel I_0(-g)+(B_\parallel\cos\alpha + B_\perp\sin\alpha)I_1(-g)\}, \tag{7.5d}$$

$$\partial m/\partial\xi = \tfrac{1}{2}(B_\parallel^2+B_\perp^2), \tag{7.5e}$$

$$\partial(g\cos\alpha)/\partial\xi = \tfrac{1}{2}(B_\parallel^2-B_\perp^2), \quad \partial(g\sin\alpha)/\partial\xi = B_\parallel B_\perp, \tag{7.5f, g}$$

where

$$m(0) = g(0) = \alpha(0) = 0. \tag{7.6a}$$

The remaining boundary conditions become

$$D_\parallel(0) = D_\perp(0) = D_\parallel(\Delta) = D_\perp(\Delta) = 0. \tag{7.6b}$$

† The type of solution sought here has some similarity with solutions presented previously by Soward (1974, §8).

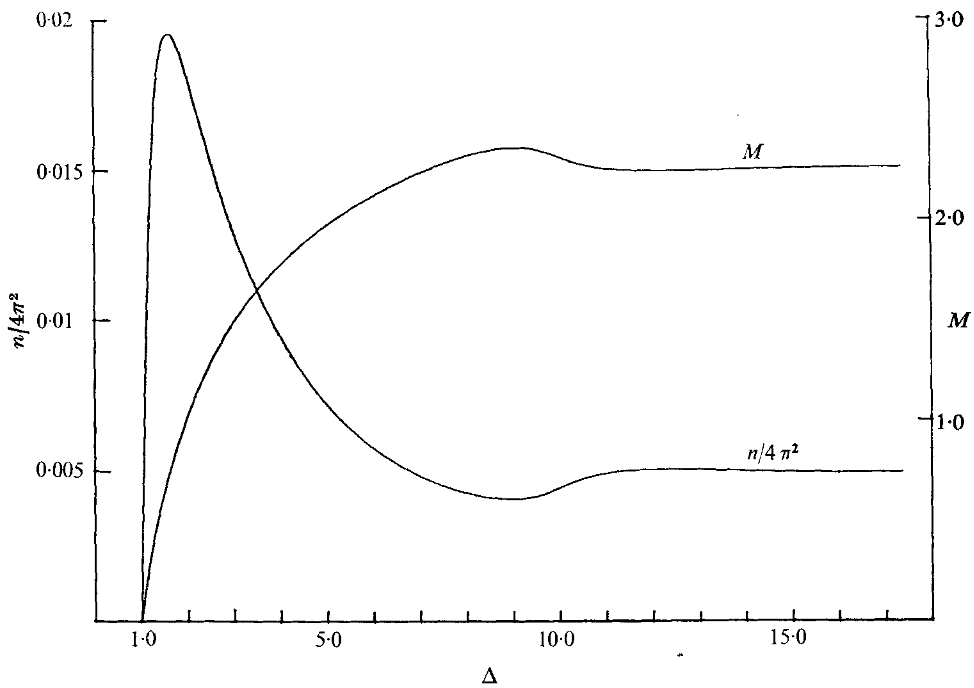

FIGURE 2. Steady solutions of the hydromagnetic dynamo investigated in §7 are characterized by the angular speed $n(\lambda/L_M^2)$ and energy per unit area $Q(\rho L_M U_0^2)\,M$ of the mean magnetic field. The two characteristics parametrized by n and M are plotted against $\Delta = L_0/L_M$, which provides a measure of the separation of the planes.

Numerical integration of these equations yielded no major surprises. The equations were integrated by the Runge–Kutta method starting at $\xi = 0$. For a given value of Δ, guesses were made of both the initial magnetic field strength $|\mathbf{B}_H(0)|$ and the angular speed n. These quantities were adjusted until the boundary condition $\mathbf{D}(\Delta) = 0$ was satisfied. For small values of $\Delta - 1$, the search was facilitated by the analytic solutions (6.13), for which good agreement with the numerical calculations was obtained. Once a solution had been located a triangulation method was used to trace the continuation of the solution to large values of Δ. In essence, a corridor in the three-dimensional Δ, M, $n/4\pi^2$ space was constructed of tetrahedrons which contained a curve C. Each point on C corresponded to a possible solution of the hydromagnetic dynamo. A more precise location of points on the curve was obtained by linear interpolation at the vertices of all triangles whose interiors were cut by the curve C. Over 1200 points were located on each of the curves in figure 2. Since each point corresponds to the solution of the differential equation a high degree of accuracy in the integration was not attempted. Checks were, however, made which suggested a 3–4 figure accuracy. Integration for moderate values of Δ appeared to be straightforward. Difficulties are necessarily encountered as $\Delta \to \infty$ for now the required solution must decay exponentially for large ξ. In fact

$$\mathbf{B}_H(\xi) \sim \mathbf{B}_H^{(1)} e^{q_1 \xi} + \mathbf{B}_H^{(2)} e^{q_2 \xi}, \tag{7.7a}$$

where q_1 and q_2 are the solutions of

$$(q/2\pi)^2 = [e^{-M} I_1(-G)]^2 - [n/2\pi q - e^{-M} I_0(-G)]^2 \quad (G = g(\infty)), \qquad (7.7b)$$

which have negative real parts. Obviously numerical integration is extremely sensitive to the exponentially growing solutions, which are not permitted.

The simplest way to interpret the solutions is to regard all parameters as fixed except for the separation L_0 of the boundaries, which is measured by the dimensionless parameter $\Delta = L_0/L_M$ (L_M fixed). For small values of Δ the length scale L_0 is imposed on the mean magnetic field [see (6.3)] and until L_0 becomes greater than L_M the magnetic induction is insufficient to overcome inevitable ohmic decay. When $L_0 = L_M$, the marginal state discussed in the previous section is achieved. As L_0 increases, the total magnetic energy $\mathscr{M}$ per unit area [see (5.15) and figure 2] begins to increase, the sheared character of the magnetic field persists and the field lines begin to rotate about the z^* axis with angular speed $n(\lambda/L_M^2)$ (see figure 2). Analytic solutions describing this behaviour near $\Delta = 1$ were, of course, given in the previous section. A curiosity of the solutions is the way the magnetic energy $\mathscr{M}$ per unit area overshoots its final asymptotic value at finite values of Δ (~ 9). A possible explanation is a sensitivity of the dynamo mechanism to the angular speed $n(\lambda/L_M^2)$. Such a sensitivity has been noted in the work of Soward (1974) on a related problem involving Bénard convection. Slight variations in the angular speed are readily accounted for by the change in the separation L_0 of the boundaries. It is reasonable therefore to suppose that the dynamo mechanism is more sensitive to $n(\lambda/L_M^2)$ than the dimensions of the space in which the magnetic field exists.

The solutions appear to settle down to their asymptotic behaviour when $\Delta \sim 12$. The numerical solution for $\Delta = 16{\cdot}6087$ is illustrated in figure 3 and is characterized by

$$M = 2{\cdot}211, \quad n/4\pi^2 = 0{\cdot}00492, \qquad (7.8a, b)$$

$$e^{-M} I_0(-G) = 0{\cdot}0856, \quad e^{-M} I_1(-G) = 0{\cdot}0417 \quad (G = g(\Delta)), \qquad (7.8c, d)$$

$$|\mathbf{B}_H(0)| = 1{\cdot}838. \qquad (7.8e)$$

Since this solution lies in the asymptotic regime it is likely to resemble the solution for $\Delta \to \infty$, corresponding to the case in which the upper boundary is absent ($L_0 \to \infty$). Unlike the case $\Delta = 1$, the magnetic field decreases in strength and ultimately decays with height at a rate $\mathrm{Re}\,(-q/L_M)$, where

$$q/2\pi = -0{\cdot}0448 \pm 0{\cdot}0975i \qquad (7.9)$$

is computed from (7.7b) on the basis of the numerical results (7.8). The significance of L_M is now apparent, for in all cases ($1 \leqslant \Delta < \infty$) L_M provides the length scale over which the mean magnetic field varies. It is remarkable that the key dynamo length L_M should be independent of the magnetic diffusivity [see (5.8b)].

The case $\Delta \to \infty$ is of particular interest as it emphasizes the mechanisms underlying the hydromagnetic dynamo. Moreover the absence of the top boundary makes the model less artificial in a geophysical or astrophysical context. To maintain the magnetic field, energy is fed into the system by exciting waves on

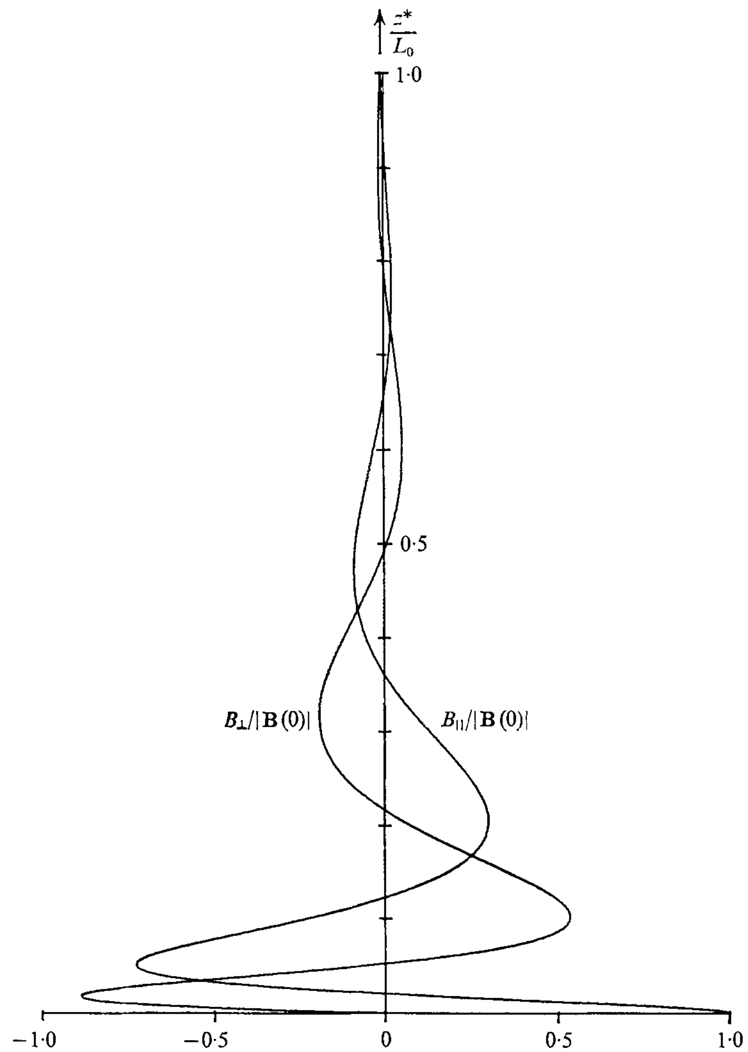

FIGURE 3. The two components $B_{\parallel}$ and $B_\perp$ of the mean magnetic field are plotted against the height above the lower plane for the steady solution that occurs when $\Delta = 16{\cdot}6087$.

the boundary $z^* = 0$. The rate of working at the boundary, which is measured by the total wave energy flux

$$\mathbf{F}_0 = 4\pi\rho U_0^3 l\mathbf{\Omega}^* \quad \left(\int_0^{2\pi} \mathbf{c}_g E d\chi \quad \text{in dimensionless variables}\right), \tag{7.10}$$

is therefore the most important characteristic of the external constraints. In view of the fact that the total wave energy density is

$$2\pi\rho U_0^2\{1+\tfrac{1}{2}(L/L_M)\,|\mathbf{B}_H(0)|^2\} \quad (L/L_M = \Omega^* U_0^2/\lambda\omega_0^{*2}) \tag{7.11}$$

at $z^* = 0$, it is curious that the energy flux $\mathbf{F}_0$ is independent of the magnetic field (at least to order ϵ). As the wave energy is transported upwards at the group velocity, a large fraction (of order 1) is lost owing to ohmic decay, a smaller fraction (of order ϵ) is transferred by wave–wave interactions to the higher harmonics, while an insignificant fraction (of order Q^{-1}) is transferred to the mean magnetic field†. The success of the dynamo, however, depends on the latter mechanism, which is manifested here by the α-effect. Since the α-effect is directly proportional to the wave energy density (kinetic only), which necessarily decreases with height [see also (5.9)], the magnetic induction process becomes less effective as z^* increases. Eventually the strength of the α-effect is insufficient to maintain the mean magnetic field and its exponential collapse ensues at a height of order L_M. Once this occurs the waves propagate unimpeded as inertial waves having an associated wave energy flux

$$\mathbf{F}_\infty = e^{-M} I_1(-G)\,\mathbf{F}_0, \tag{7.12}$$

while the strength of the α-effect remains constant. Finally, it may be speculated that the rotation of the mean magnetic field lines about the z^* axis on the free decay time scale L_M^2/λ can be attributed to the non-uniformity of the α-effect.

It is instructive to reassess the energetics of the system on a global basis. The energy density of the mean magnetic field, of order $Q\rho U_0^2$, is larger than the wave energy density by a factor Q. The former, however, is continually lost at a rate of order λ/L_M^2 but is replenished at a rate of order $Q\lambda/L_M^2$ through the continual conversion of wave energy. The latter decreases at a rate of order $l\Omega^*/L_M$ (larger than the former rate by a factor L_M/l, of order Q) by direct ohmic dissipation. Thus, whereas the dynamo could be called efficient on the basis of the ratio of mean magnetic energy density to wave energy density, the conversion of wave energy into mean-field energy is clearly inefficient.

It was argued in §4 that low frequency oscillations provide the most significant contribution to the α-effect. The reasoning was essentially based on the weak-field solutions of §5, for which the criterion $\Delta \geqslant 1$ was necessary for dynamo action. Since Δ is inversely proportional to the square of the frequency ω_0^*, the criterion can always be satisfied provided that ω_0^* is sufficiently small. For the fully developed hydromagnetic dynamo, which occurs in the limit $\Delta \to \infty$, the energy density of the mean magnetic field is insensitive to changes in ω_0^*. The length scale L_M of the mean magnetic field is however proportional to ω_0^{*2} and consequently the magnetic energy $\mathscr{M}$ per unit area decreases with decreasing ω_0^*. If the size of $\mathscr{M}$ is adopted as a criterion for selecting the most potent modes it is now no longer clear that restricting attenuation to low frequency modes is reasonable for the unbounded case $L_0 \to \infty$.

The finite wave energy flux $\mathbf{F}_\infty$ carried off by the inertial waves to infinity raises some awkward questions. Suppose that a seed magnetic field is introduced at large z^*/L_M. Then provided its length scale is sufficiently large and boundary conditions are ignored the analysis in I certainly indicates that the seed magnetic field will grow. The neglect of boundary conditions, when computing the ultimate steady state, is evidently unreasonable. On the other hand the simplicity of the

† Even at this level energy transfer to the mean flow is unimportant. For, as in II §7, it can be argued that no mean flow is created either by the Lorentz force $(\nabla \times \mathbf{B}) \times \mathbf{B}$ or the Reynolds stresses $\langle u_{0i} u_{2j} \rangle$ and $\langle b_{0i} b_{2j} \rangle$.

argument seriously suggests possible instability in the model. The non-zero wave energy flux $\mathbf{F}_\infty$ may be traced to the total magnetic energy $\mathscr{M}$ [see (7.12)], whose finite value is a direct consequence of the exponential decay of magnetic field. Indeed a solution without exponential decay is only compatible with (7.7*b*) when the magnetic field is stationary ($n = 0$). But, since the α-effect necessarily decays and in view of the speculation below (7.12), this stationary state is unlikely. Therefore the present model, with its finite skin depth, appears reasonable and may be stable.

Serious fundamental difficulties are raised by adopting the limit $L_0 \to \infty$ in its literal sense and stem from the time, of order $L_0/l\Omega^*$, required for the wave energy to be transported across the gap. Consider the effect of relaxing the original assumption that L_0 is of order L. When $L_0/l\Omega^*$ becomes comparable with the time scale $\epsilon^{-1}Q/\Omega^*$ for significant energy interchange between modes to occur owing to nonlinear wave interactions, first-order smoothing can no longer be justified everywhere in the interval $0 \leqslant z^* \leqslant L_0$. The breakdown occurs when L_0 is of order $\epsilon^{-1} L$. The sizes of both the ratios Δ and L_0/L are, therefore, restricted by the necessity of not violating any previous approximations based on the sizes of ϵ and Q. The dynamo model attributed to the limit $L_0 \to \infty$ can then be justified only under the triple inequality

$$1 \gg \Delta^{-1} \gg \epsilon \gg Q^{-\frac{1}{2}} \quad (L/L_M = O(1)). \tag{7.13}$$

The conclusions of the previous paragraphs are thus valid only under these stringent limitations.

The author wishes to thank Professor P. H. Roberts and Dr A. Davey for some helpful comments.

Appendix A

A list is provided of the principal properties of the 'helicity projection operator'

$$M^s_{ij}(\mathbf{k}) = \tfrac{1}{2}\{(\delta_{ij} - \hat{k}_i\hat{k}_j) + is\epsilon_{inj}\hat{k}_n\},$$

where $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$.

$$\delta_{ij} = M^s_{ij} + M^{-s}_{ij} + \hat{k}_i\hat{k}_j, \tag{A 1}$$

$$M^s_{ij} = M^{-s}_{ji}, \quad M^s_{ij}M^{s'}_{jk} = \delta_{ss'}M^s_{ik}, \qquad \text{(A 2,) (A 3)}$$

$$\epsilon_{ijk}M^s_{kl} = is(M^s_{jl}\hat{k}_i - M^s_{il}\hat{k}_j), \quad \epsilon_{ijk}\hat{k}_jM^s_{kl} = -isM^s_{il}, \qquad \text{(A 4), (A 5)}$$

$$\epsilon_{ijk}M^s_{jl}M^{s'}_{mk} = -is\delta_{ss'}\hat{k}_iM^s_{ml}, \quad M^s_{ij}M^s_{kl} = M^s_{il}M^s_{kj}. \qquad \text{(A 6), (A 7)}$$

The helicity property of M^s_{ij} is demonstrated by (A 5), while its projection property results from (A 3). Incidently (A 1) provides a decomposition of the identity matrix into three projection operators having the property that the product of any distinct pair vanishes.

The property (A 7) is especially useful as it defines the shape of the energy spectrum tensor [see (2.31) and (3.9)]. The spectrum tensor is thus completely defined by the energy spectrum (a scalar quantity).

Appendix B

The definitions $\mathbf{k} = \nabla\theta$ and $\omega = -\partial\theta/\partial t$ yield the well-known identity

$$\partial k_i/\partial\tau = -\partial\omega/\partial X_i, \tag{B 1}$$

where $\mathbf{k} = \mathbf{k}(\mathbf{k}_0; \mathbf{X}, \tau)$ and $\omega = \omega(\mathbf{k}; \mathbf{X}, \tau)$. The equation is differentiated with respect to k_{0j} (an operation which commutes, here, with $\partial/\partial\tau$ and $\partial/\partial X_i$ as $\mathbf{k}_0$ is fixed with respect to the latter operators) and yields

$$\frac{\partial}{\partial\tau}\left(\frac{\partial k_i}{\partial k_{0j}}\right) = -\frac{\partial}{\partial X_i}\left([\mathbf{c}_g]_l \frac{\partial k_l}{\partial k_{0j}}\right). \tag{B 2}$$

Since

$$\sigma^{-1} = \frac{1}{3!}\epsilon_{ijk}\epsilon_{pqr}\frac{\partial k_i}{\partial k_{0p}}\frac{\partial k_j}{\partial k_{0q}}\frac{\partial k_k}{\partial k_{0r}}, \tag{B 3}$$

routine manipulation leads to the continuity equation,

$$\partial(\sigma^{-1})/\partial\tau + \nabla.(\sigma^{-1}\mathbf{c}_g) = 0. \tag{B 4}$$

The material derivative

$$D/D\tau \equiv \partial/\partial\tau + \mathbf{c}_g.\nabla \tag{B 5}$$

plays a key role in the analysis in § 3. Here differentiations refer to fixed $\mathbf{k}_0$. The material derivative for fixed $\mathbf{k}$ introduced at the end of § 3 is determined by the identity

$$\frac{D}{D\tau} = \left[\frac{D}{D\tau}\right]_{\mathbf{k}\ \text{fixed}} + \frac{Dk_i}{D\tau}\frac{\partial}{\partial k_i} \tag{B 6}$$

and the well-known identity

$$\frac{Dk_i}{D\tau} = -\left[\frac{\partial\omega}{\partial X_i}\right]_{\mathbf{k}\ \text{fixed}} \tag{B 7}$$

(see, for example, Bretherton 1970, p. 74).

Appendix C

The effects (see (*a*) and (*b*) below) highlighted by the general theory are not investigated in I. The following discussion suggests that neglect of these effects cannot be justified.

(*a*) *Inertial wave interactions*

When two waves of frequency ω_1 and ω_2 interact, the resulting Reynolds stresses excite two new modes of frequency $\omega_1 \pm \omega_2$. Moreover, this new mode, whose amplitude is of order $R_0 U_0$, can interact with one of the original inertial waves causing a resonant excitation of the other. An order-of-magnitude estimate shows that the time scale for the resonance is $t_r^* = (R_0^2\Omega^*)^{-1}$.

Equations governing the development of the energy spectrum for weakly interacting nonlinear random waves have been derived by Benney & Saffman (1966) and Benney & Newell (1969). Their analysis indicates that the energy spectrum evolves on the time scale t_r^* despite the fact that energy transfer is dominated by resonant triad interactions as opposed to the quartic interaction described above. Their analysis hinges on model equations which are simpler than the equations governing the inertial waves. The added complication occurs in the latter case because of the relatively large region in wave vector space for which the frequency of inertial waves is small ($\hat{\mathbf{k}}.\mathbf{\Omega} \ll 1$). This inevitably causes non-uniformities in the expansion procedures. Despite the mathematical difficulties it is clear that the influence of the low frequency modes cannot be to lengthen the time scale of energy transfer.

In I dynamo action was first investigated on the basis of a kinematic theory in which the flow was not influenced by Lorentz forces. A dominant mode for the mean magnetic field was found to emerge. Fortunately, except for its time dependence, this mode satisfied the equations governing the subsequent nonlinear development, when the magnetic field was no longer weak. The hydromagnetic dynamo problem was then resolved by determining an equation for the magnetic energy.

Of crucial importance is the time scale on which dynamo action occurs. The e-folding time t_e^*, determined by equation (4.16) in I, for the growth of the magnetic field in the linear regime is

$$\left.\begin{array}{ll} R_0^{-4}\Omega^{*-1} & \text{when} \quad Q = O(1), \\ R_0^{-2}(R_0^2 Q)^{-1}\Omega^{*-1} & \text{when} \quad Q \gg 1. \end{array}\right\} \tag{C 1}$$

These estimates must certainly provide lower bounds on the dynamo time scale. Since the analysis of I is valid only when $R_0^2 Q \ll 1$, it follows that

$$t_e^* \gg t_r^*. \tag{C 2}$$

Consequently the energy spectrum is likely to evolve, because of nonlinear wave interactions, much faster than the magnetic field. It is therefore reasonable to suppose that the neglect of wave–wave interactions may have serious consequences.

(*b*) *Energy transport at the group velocity*

The evolution of the waves in I is based on the assumption that the magnetic field is uniform and constant. Therefore the energy equation (2.32) follows immediately from (2.16). The slow decay rate

$$Q^{-1}\omega_i = \frac{(k^2/Q)(\mathbf{k}.\mathbf{B})^2}{4(\hat{\mathbf{k}}.\mathbf{\Omega})^2 + (k^2/Q)^2} \tag{C 3}$$

is, however, calculated on the assumption that the magnetic field is weak and consequently differs from (2.18), which is derived on the basis of large Q.

The derivation of e in I takes no account of advection of wave energy at the group velocity. This procedure can only be justified *a posteriori* if

$$\mathbf{c}_g(\sim l\Omega^*) \ll L_e/t_e^*, \tag{C 4}$$

where L_e is the length scale of the mean magnetic field. As before, equation (4.16) in I can be used to give the order-of-magnitude estimate

$$L_e/t_e^* = R_0^2(l\Omega^*), \tag{C 5}$$

valid for all values of Q. Since R_0 is small, (C4) and (C5) are incompatible and, moreover, indicate that the advection terms in (2.32) and (3.13) are in fact larger than the time-derivative term by an order of magnitude.

The effect of strong advection at the group velocity suggests a slight modification to the analysis of I. To leading order e is independent of Z, so that the equation for the Z average of e, say $\bar{e}$, is

$$\partial\bar{e}/\partial\tau = -2\bar{\omega}_i\bar{e}. \tag{C 6}$$

A uniformly valid first approximation to the energy $\bar{e}$ is, therefore,

$$e = e_0 \exp\left[-\int_0^\tau 2\bar{\omega}\,d\tau\right]. \tag{C 7}$$

REFERENCES

ACHESON, D. J. 1972 The critical level for hydromagnetic waves in a rotating fluid. *J. Fluid Mech.* **53**, 401.

BENNEY, D. J. & NEWELL, A. C. 1969 Random wave closures. *Studies in Appl. Math.* **48**, 29.

BENNEY, D. J. & SAFFMAN, P. G. 1966 Nonlinear interactions of random waves in a dispersive medium. *Proc. Roy. Soc.* A **289**, 301.

BRAGINSKIĬ, S. I. 1964*a* Self excitation of a magnetic field during the motion of a highly conducting fluid. *Zh. Eksp. Teor. Fiz.* **47**, 1084. (Trans. *Sov. Phys. J. Exp. Theor. Phys.* **20** (1965), 726.)

BRAGINSKIĬ, S. I. 1964*b* Theory of the hydromagnetic dynamo. *Zh. Eksp. Teor. Fiz.* **47**, 2178. (Trans. *Sov. Phys. J. Exp. Theor. Phys.* **20** (1965), 1462.)

BRAGINSKIĬ, S. I. 1964*c* Magnetohydrodynamics of the earth's core. *Geomag. Aeron.* **4**, 732.

BRAGINSKIĬ, S. I. 1967 Magnetic waves in the earth's core. *Geomag. Aeron.* **7**, 1050.

BRETHERTON, F. P. 1970 The general linearized theory of wave propagation. In *Mathematical Problems in Geophysical Sciences* (ed. W. H. Reid), p. 61. Am. Math. Soc.

BUSSE, F. H. 1973 Generation of magnetic fields by convection. *J. Fluid Mech.* **57**, 529.

CHANDRASEKHAR, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford University Press.

CHILDRESS, S. & SOWARD, A. M. 1972 Convection driven hydromagnetic dynamo. *Phys. Rev. Lett.* **29**, 837.

GREENSPAN, H. P. 1968 *The Theory of Rotating Fluids*. Cambridge University Press.

GREENSPAN, H. P. 1969 On non-linear interaction of inertial modes. *J. Fluid Mech.* **36**, 257.

GUBBINS, D. 1974 Dynamo action of isotropically driven motions of a rotating fluid. *Studies in Appl. Math.* **53**, 157.

KAWAHARA, T. 1973 The derivative-expansion method and nonlinear dispersive waves. *J. Phys. Soc. Japan*, **35**, 1537.

LANDAHL, M. T. 1972 Wave mechanics of breakdown. *J. Fluid Mech.* **56**, 775.

LEHNERT, B. 1954 Magnetohydrodynamic waves under the action of the Coriolis force I. *Astrophys. J.* **119**, 647.

MOFFATT, H. K. 1970 Dynamo action associated with random inertial waves in a rotating conducting fluid. *J. Fluid Mech.* **44**, 705.

MOFFATT, H. K. 1972 An approach to a dynamic theory of dynamo action in a rotating conducting fluid. *J. Fluid Mech.* **53**, 385.

RÄDLER, H.-K. 1968 On the electrodynamics of a conducting fluid in turbulent motion. I. The principles of mean field electrodynamics. *Z. Naturforsch.* **23***a*, 1841. (Trans. in Roberts & Stix 1971.)

ROBERTS, P. H. & SOWARD, A. M. 1975 A unified approach to mean field electrodynamics. *Astrom. Nachr.* **296**, 49.

ROBERTS, P. H. & STIX, M. 1971 The turbulent dynamo. *N.C.A.R., Boulder, Colorado, Tech. Note,* 1A–60.

SOWARD, A. M. 1972 A kinematic theory of large magnetic Reynolds number dynamos. *Phil. Trans.* A **272**, 431.

SOWARD, A. M. 1974 A convection-driven dynamo I. The weak field case. *Phil. Trans.* A **275**, 611.

SOWARD, A. M. & ROBERTS, P. H. 1975 Recent developments in dynamo theory. *Magnetnaya Gidnodynamicka.* To appear.

WHITHAM, G. B. 1965 A general approach to linear and non-linear dispersive waves using a Lagrangian. *J. Fluid Mech.* **22**, 273.